tribution shown here although the situation previously could be described as murky.

We conclude that there is a variation of the parameter B with Q^2 and any reformulation of the theory should take this into account.

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Reactive Content of the First-Order Optical Potential

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It is shown that the total reaction cross section appropriate to the optical potential for scattering of a projectile by a target of uncorrelated particles (nucleons) can be written to a good approximation as a sum of exclusive cross sections for n-nucleon knockout. The association made by some authors of the reactive content of this optical potential with single-nucleon knockout is shown to apply to the inclusive-reaction cross section only.

The subject of this Letter is the connection between the optical potential for elastic scattering of a particle from a complex target, such as a nucleus, and the cross sections for inelastic reactions induced by the incident particle. In general, one may relate the total reactive cross section σ_r, defined as the difference between the total and elastic cross sections $\sigma_r = \sigma_T - \sigma_{\rm el}$, to the optical potential $\mathfrak{U}$, by[1]

$$\sigma_r = C\langle \varphi_0^{(+)}| \mathfrak{U}^\dagger - \mathfrak{U} | \varphi_0^{(+)}\rangle \qquad (1)$$
$$= -2iC\,\mathrm{Im}\langle\varphi_0^{(+)}|\mathfrak{U}|\varphi_0^{(+)}\rangle ,$$

where $C = -i(2\pi)^3 u^{-1}$, with u the projectile velocity, and $\varphi_0^{(+)}$ is the elastic wave which satisfies the optical scattering equation with the potential $\mathfrak{U}$. Equation (1) is an expression of the unitary relation between the flux lost from the elastic channel and the imaginary part of $\mathfrak{U}$.

The question of the reactive content may be posed: Assuming that some approximate theory is given for $\mathfrak{U}$, what is implied about the nonelastic reactions; for example, which reaction channels are included in the theory and how are the partial cross sections to these channels to be calculated? This question has been discussed recently by a number of authors[2-6] in the context of multiple-scattering theory, mostly concerning the first-order approximation

$$\mathfrak{U} \simeq \langle \textstyle\sum_i t_i\rangle \qquad (2)$$

in terms of the scattering of the projectile from individual nucleons, where the expectation value is over the target ground state. The first correction to (2) is due to correlations among the nucleons. The form (2) defines a class of theories (appropriate to uncorrelated targets) which differ in what is used for t_i. It is generally agreed that inelastic scattering in which one target nucleon is certainly implied by (2). This has been studied in detail in a three-body model, by Tandy, Redish, and Bolle.[2] Application of their results to pion-nucleus scattering has been made by Thomas and Landau[3]; further discussion has been given by Eisenberg,[4] and by Ernst and Thaler,[5] who give unitary relations for higher-order approximations to $\mathcal{U}$.

In this paper we address two related questions which are raised by the published discussion, and on which we have obtained new results. The questions are whether the inelastic excitation of single nucleons exhausts the reactive content of first-order optical potentials of the form (2), and what is the relation of the unitary equation (1) in such theories to the distorted-wave (DW) theory of inelastic scattering. Our answers to these questions take the form of sum rules for inelastic cross sections, in which the *inclusive* reactive cross section (1) is expressed as a sum of *exclusive* cross sections for each of which a given number (n) of nucleons is excited (e.g., knocked out). These exclusive cross sections are to be calculated in a form of DW approximation.

We consider a model nuclear target with A nucleons bound in a single-particle potential and no mutual interaction, therefore uncorrelated. We shall take the nucleons to be distinguishable (as in Watson's multiple-scattering theory) and for simplicity assume that there is only one bound orbital for each nucleon, so that any excitation is into the continuum. The projectile is an elementary particle distinguished from the nucleons, e.g., a meson. The system obeys a Schrödinger equation with Hamiltonian $H = H^0 + V$, where

$$H^0 = K_p + \sum_{i=1}^{A} H^N(i),$$

with K_p the kinetic energy of the projectile and $H^N(i)$ the single-particle nucleon Hamiltonian, where

$$V = \sum_{i=1}^{A} v_i$$

is the interaction of the projectile with the individual nucleons. The optical potential $\mathcal{U}$ for elastic scattering of the projectile may be expressed in the form given by Feshbach,[7]

$$\mathcal{U} = P_0 U_0 P_0,$$

with

$$U_0 = V + VQ[Q(E^+ - H^0 - V)Q]^{-1}V, \tag{3}$$

where P_0 is the projection operator on the vector space of the entire system onto the elastic-channel (ground-state target) space, and Q that onto the inelastic space: $P_0 + Q = 1$. We use E^+ to denote the usual limit $E + i\eta$, $\eta \to 0^+$, with E the total energy. We further decompose the Q space by

$$Q = \sum_{n=1}^{A} P_n,$$

where P_n projects onto the subspace with n excited nucleons; this defines the inelastic channel with n nucleons in the continuum. These subspaces are connected by V, with the selection rule $P_n V P_m = 0$ unless $m = n, n \pm 1$, because V is a one-nucleon operator. This can be used to decompose (3) into a continued-fraction form,

$$U_0 = V + VP_1 G_1 P_1 V, \tag{4}$$

where G_1 couples P_1 to P_2, and so on, as we shall see. But first we approximate (4) by inserting $V = \sum_i v_i$, and dropping $i \neq i'$ terms in (4), so that U_0 takes the form

$$U_0 \cong \sum_i^A [v_i + v_i P_1 \tilde{G}_1(i) P_1 v_i], \tag{5a}$$

$$\tilde{G}_1(i) = [P_1\{E^+ - H^0 - v_i - U_1(i)\}P_1]^{-1}, \tag{5b}$$

$$U_1(i) \cong \sum_j{}' [v_j + v_j P_2 \tilde{G}_2(i,j) P_2 v_j], \tag{5c}$$

where the sum in (5c) excludes $j = i$. To obtain (5c) we have dropped $j \neq j'$ terms, as in (5a). The form of Eqs. (5) follows from the fact that v_i only excites the ith nucleon; to reach P_2, a second nucleon ($k \neq i$) must be struck. This expansion can be continued to all channels n, but first we write (5a) in t-matrix form, as in (2):

$$\begin{aligned} &U_0 \cong \sum_i^A t_i, \quad t_i = v_i + v_i G_1(i) t_i, \\ &G_1(i) = P_1[P_1\{E^+ - H^0 - U_1(i)\}P_1]^{-1}P_1. \end{aligned} \tag{6}$$

Note that the t matrix defined in (6) is not the free-projectile–nucleon t matrix, since H^0 includes the nucleon binding potential, and $U_1(i)$ is an average interaction of the projectile with the $A-1$ bound nucleons ($j \neq i$) (local-field correction). Since there are no pair correlations in our model, U_0 has no terms of second order (in t_i), so that the approximation leading to (5a) and (5c) involves neglecting "reflection" terms, which are multiple-scattering effects of fourth order (and higher).

We may complete the expression of (5) and (6) by a set of recursive relations:

$$U_n(i_1, \ldots, i_n) = \sum_j{}' t_j(i_1, \ldots, i_n), \tag{7a}$$

$$t_j(i_1, \ldots, i_n) = v_j + v_j G_{n+1}(i_1, \ldots, i_n, j) t_j(i_1, \ldots, i_n), \tag{7b}$$

$$G_{n+1}(i_1, \ldots, i_n, j) \cong P_{n+1}[P_{n+1}\{E^+ - H^0 - U_{n+1}(i_1, \ldots, i_n, j)\}P_{n+1}]^{-1}P_{n+1}, \tag{7c}$$

where the summation in (7a) excludes $j = i_1, \ldots, i_n$. The last expression (7c) involves an additional approximation of neglecting the interaction terms

$$\sum_{k=i_1}^{i_n} v_k$$

between the projectile and the previously excited (knocked-out) nucleons. [Again, $j \neq j'$ terms are neglected in (7a).]

We may calculate the reactive cross section σ_r for our model, using (6) and (1):

$$\sigma_r = C\langle \varphi_0^{(+)} | \Delta U_0 | \varphi_0^{(+)} \rangle = C\sum_i \langle \varphi_0^{(+)} | \Delta t_i | \varphi_0^{(+)} \rangle, \tag{8}$$

where $\Delta A \equiv A^\dagger - A$; note that $P_0\varphi_0^{(+)} = \varphi_0^{(+)}$. We evaluate Δt_i from (6) and (7) using an operator relation[2] which gives

$$\Delta A = (1 + CA)^\dagger \Delta B (1 + CA) + A^\dagger \Delta C A \tag{9}$$

for operators A, B, and C such that $A = B + BCA$. We find

$$\Delta t_j(i_1, \ldots, i_n) = t_j^\dagger(i_1, \ldots, i_n) \Delta G_{n+1}(i_1, \ldots, i_n, j) t_j(i_1, \ldots, i_n), \tag{10a}$$

$$\Delta G_{n+1}(i_1, \ldots, i_n, j) = 2\pi i(1 + U_{n+1}G_{n+1})^\dagger \delta(E - H^0) P_{n+1}(1 + U_{n+1}G_{n+1}) + G_{n+1}^\dagger \sum_k{}' \Delta t_k(i_1, \ldots, i_n, j) G_{n+1}, \tag{10b}$$

where we have suppressed some particle indices in (10b). To obtain (10b) we write $G_n = \mathcal{G}_n + \mathcal{G}_n U_n G_n$, $\mathcal{G}_n = [P_n(E^+ - H^0)P_n]^{-1}$, $\Delta \mathcal{G}_n = 2\pi i \delta(E - H^0) P_n$, and use (9). Substituting (10) in (8) recursively, we obtain the summed form

$$\sigma_r = \sum_{n=1}^{A} \sigma(n), \tag{11a}$$

where

$$\sigma(n) = [(2\pi)^4/u] \sum_{i_1, \ldots, i_n} \langle \varphi_0^{(+)} | t_{i_1}^\dagger [G_1^\dagger(i_1) t_{i_2}(i_1)] \cdots (1 + U_n G_n)^\dagger \times P_n \delta(E - H^0) P_n (1 + U_n G_n) \cdots [t_{i_2}(i_1) G_1(i_1)] t_{i_1} | \varphi_0^{(+)} \rangle \tag{11b}$$

$$= [(2\pi)^4/u] \sum_{i_1, \ldots, i_n} \int |\langle \varphi_n^{(-)}(i_1, \ldots, i_n)| T | \varphi_0^{(+)} \rangle|^2 \delta(E - H_n^0) dk_n. \tag{11c}$$

This is the principal result of this paper, which is interpreted as follows. The reaction cross section (8) for the first-order optical potential $U_0 = \sum_i t_i$ (6) may be written as a sum (11a) of *exclusive* cross sections into the nth channel with n nucleons excited. The $\sigma(n)$ involve a series of inelastic collisions, knocking out nucleons $i_1, \ldots, i_n$ (11b), which may be written in DW form (11c), where the transition operator is given by

$$T = t_{i_n} G_{n-1} t_{i_{n-1}} \cdots G_1 t_{i_1}, \tag{11d}$$

with some indices suppressed. We indicate the integration over the momenta of n nucleons plus projectile by dk_n. The final wave $\varphi_n^{(-)}$ has each outgoing nucleon distorted by the binding potential, and the projectile by the potential $U_n(i_1, \ldots, i_n)$ due to the residual target. Similar potentials appear in the propagators $G_{n-1}, \ldots, G_1$ in (11d). There is no interference between different orders of collision among the n nucleons in (11b) and (11c); this is lost in the neglect of $i \neq i'$ terms in (5) and (7).

So we have shown that the total reaction cross section σ_r corresponding to the first-order optical potential (6) is given by the sum (11a) of exclusive cross sections $\sigma(n)$, each of which is calculated in a DW approximation, using the t_j of (7) for inelastic scattering, and U_m $(m \leq n)$ for distortion of the projectile wave. Our t_j, although single-scattering operators (i.e., can excite only nucleon j), are more complicated than conventionally used in multiple-scattering theory.[8] However, the specific form (7b) was required only in order to derive Eqs. (11). For application of (11) it may well be possible to compute with simpler approximate forms of t_j (e.g., impulse approximation), in which case the sum rule (11a) still applies, and the $\sigma(n)$ become conventional DW impulse-approximation cross sections.

If such simplifying approximations can be made for the t_i when calculating U_0 in (6), then we may also obtain an alternative expression for σ_r, directly from (8). For example, if we can ignore the "local field" potential $U_1(i)$ in t_i in the evaluation of (8), we obtain

$$\sigma_r \cong \frac{(2\pi)^4}{u} \times \sum_i \langle \varphi_0^{(+)} | t_i^\dagger P_1 \delta(E - H^0) P_1 t_i | \varphi_0^{(+)} \rangle . \quad (12)$$

This formula is equivalent to the result of Tandy, Redish, and Bollé,[2] which expresses σ_r in terms of a cross section for knockout of a single nucleon, but with *no distortion of the outgoing projectile* [since U_1 does not appear in (12)]. Comparison with (11) and its interpretation now makes the meaning of (12) clear: This is the *inclusive* cross section for inelastic scattering, corresponding to the first-order optical potential. Note that the approximation leading to (12) requires that $U_1(i)$ not contribute to U_0 through t_i; this does not require that $U_1(i)$ be zero or small, but that t_i not be sensitive to $U_1(i)$. This condition would hold *a fortiori* if the *impulse* approximation for t_i were valid, e.g., at high energies. The final projectile wave in (12) is not distorted because that is inappropriate for evaluating an *inclusive* cross section. On the other hand, the *exclusive* cross section for knockout of a single nucleon, $\sigma(1)$ in (11), is evaluated with optical distortion of the projectile by $U_1(i)$ [in $\varphi_1^{(-)}(i)$]. [A similar point has been made[9] for the eikonal approximation to σ_r and $\sigma(1)$.] Any flux removed from the P_1 channel by $U_1(i)$ is fed into the P_2 channel, and so on.

The exclusive sum rule (11) for σ_r must hold whenever (12) is valid since the assumptions required for the former are more general than for (12). However, (11) is only appropriate under the assumption that the target is uncorrelated, for which the first-order approximation to the optical potential U_0 is valid. Extensions of the considerations of this paper to correlated targets is complicated by the fact that the decomposition of Q into channels P_n no longer leads to the continued-fraction form (5). However, Pauli correlations (antisymmetry) may be included in the present formulation, at small extra cost. (Similarly, we may include bound excited states in the P_n.)

In the high-energy approximation, and for a large target, the results above can be further simplified, and related to the methods of Glauber.[10] More details will be given in a longer report.

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