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Macroscopic Electron Lattice on the Surface of Liquid Helium

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(Received 4 April 1979)

A nonlinear theory of the capillary waves accounts for the macroscopic lattice formation on the electron-charged surface of liquid helium observed in a recent experiment. The lattice allows both shear and compressional lattice waves to propagate.

There has been a great deal of interest in systems consisting of a layer of electrons on the surface of liquid helium. Recently, Wanner and Leiderer[1] found experimentally that a macroscopic dimple lattice was created at an electron-charged interface between ^{3}He and ^{4}He. The theory presented here accounts for most of the observed results. A preliminary discussion of the present problem was given by Gor'kov and Chernikova.[2]

Consider the liquid helium filling a volume $-z_0 \leq z \leq 0$. The surface of the liquid is in the x-y plane ($z=0$) when it is not disturbed. The surface of the liquid is charged with electrons. The electric field (E_+ at $z>0$, and E_- at $z<0$) is applied in the vertical, z, direction to bind the electrons to the surface.

The nonlinear dispersion relation of the capillary waves of frequency ω, as a function of the wave vector $\vec{k}$, is given by[3,4]

$$\omega^2(\vec{k}) = g|\vec{k}| - (1/4\pi\rho)(E_+^2+E_-^2)|\vec{k}|^2 + (\tau/\rho)|\vec{k}|^3, \qquad (1)$$

where ρ is the density of the liquid, g the gravity constant, and τ the surface tension. The Rayleigh-Taylor instability grows, i.e., $\omega^2<0$ near $|\vec{k}|=k_0$, when $\frac{1}{2}(E_+^2+E_-^2) > 4\pi g\rho/k_0$, where $k_0^2=\rho g/\tau$. The nonlinear behavior of this instability will be analyzed here.

It is assumed that the liquid is incompressible and that the electrons redistribute instantaneously in such a way that the electrostatic potential is always constant along the surface. The large electron mobility ensures that latter assumption. By employing the basic equations and the procedure described by Mima and Ikezi,[5] the equation governing the surface deviation, $z=a(x,y,t)$, is found to be

$$\begin{aligned} a_{tt} = {} & -g\frac{P}{T}a + \frac{E_+^2+E_-^2}{4\pi\rho}P^2a + \frac{(E_+^2-E_-^2)}{8\pi\rho}\frac{P}{T}\left[2PTaPTa - 2aP^2a + (PTa)^2 + a_x^2 + a_y^2\right] \\ & + \frac{E_+^2+E_-^2}{4\pi\rho}\frac{P}{T}\Big[PT\left(-\tfrac{1}{2}a^2P^2a + aPTaPTa - aP^2aPTa\right) + \tfrac{1}{2}a^2P^3Ta + (a_x^2+a_y^2)PTa \\ & \qquad + \bigl(PTa\bigr)\left(PTaPTa - aP^2a\right)\Big] \\ & + \frac{\tau}{\rho}\frac{P}{T}\left[a_{xx}\left(1-\tfrac{3}{2}a_x^2-\tfrac{1}{2}a_y^2\right) + a_{yy}\left(1-\tfrac{3}{2}a_y^2-\tfrac{1}{2}a_x^2\right) - 2a_{xy}a_xa_y\right]. \end{aligned} \qquad (2)$$

Here, P and T are operators defined by

$$P^2 = -\nabla_\perp^2 = -(\partial^2/\partial x^2 + \partial^2/\partial y^2),$$

and $T=\coth(z_0P)$, the suffixes x, y, and t indicate partial derivatives with respect to those variables, and the P and T enclosed in $\bigl(\,\bigr)$ do not operate outside of the heavy parentheses. The linear terms of this equation give us the dispersion relation (1). The nonlinear terms which are important for discussing the Rayleigh-Taylor instability are retained.

Because the parameter range of interest is $\frac{1}{2}(E_+^2+E_-^2) \sim 4\pi g\rho/k_0$ and $|\vec{k}| \sim k_0$, the waves of the form $k_0 a = \sum_j F_j(\vec{r},t)\exp(i\vec{k}_j\cdot\vec{r})$ are considered. The F_j's are slowly varying functions of $\vec{r}$ and t, and $|\vec{k}_j| = k_0$. Of the many unstable waves, let us consider three waves F_1, F_2, and F_3, which satisfy $\arccos(\vec{k}_1\cdot\vec{k}_2/k_1k_2) = \arccos(\vec{k}_2\cdot\vec{k}_3/k_2k_3) = \frac{1}{3}\pi$. Only when this relation is satisfied, do the wave vectors meet the matching condition $\vec{k}_1 + \vec{k}_3 = \vec{k}_2$ and the waves parametrically couple in the second-order term of Eq. (2). The combinations $(\vec{k}_2, \vec{k}_3, -\vec{k}_1)$, $(\vec{k}_3, -\vec{k}_1, -\vec{k}_2)$, etc., also satisfy the matching condition. Therefore, we need to consider six waves

$$k_0 a = \sum_{j=1}^{3}[F_j\exp(i\vec{k}_j\cdot\vec{r}) + F_{-j}\exp(-i\vec{k}_j\cdot\vec{r})], \quad (3)$$

which couple and grow together. Substitution of Eq. (3), together with nonlinearly driven waves, into Eq. (2) yields six coupled equations in the deep-liquid limit, $z_0\to\infty$:

$$(gk_0)^{-1}[F_{1tt} - v_g^{\,2}(\hat{k}_1\cdot\nabla)^2F_1] = 2\alpha F_1 + 3\beta F_2F_{-3} - [\gamma_1F_1F_{-1} + \gamma_2(F_2F_{-2} + F_3F_{-3})]F_1,$$
$$(gk_0)^{-1}[F_{2tt} - v_g^{\,2}(\hat{k}_2\cdot\nabla)^2F_2] = 2\alpha F_2 + 3\beta F_1F_3 - [\gamma_1F_2F_{-2} + \gamma_2(F_1F_{-1} + F_3F_{-3})]F_2, \text{ etc.}, \quad (4)$$

where $v_g^{\,2} = g/k_0$, $\alpha = \frac{1}{2}(E_+^{\,2} + E_-^{\,2})/(4\pi g\rho/k_0) - 1$, $\beta = (E_+^{\,2} - E_-^{\,2})/(E_+^{\,2} + E_-^{\,2}) < 0$, $\gamma_1 = \frac{5}{2} - 8\beta^2$, $\gamma_2 = 8(2 - \sqrt{3}) - 1 - (57\sqrt{3} - 72)\beta^2/(4\sqrt{3} - 6)$, and $\hat{k}_j = \vec{k}_j/k_0$.

A solution is now sought of the form

$$F_j = R(t)\exp[-i\theta_j(\vec{r})]. \quad (5)$$

If $\hat{k}_j\cdot\nabla\theta_j = 0$ and F_{-j} is the complex conjugate of F_j, then

$$\theta_1 + \theta_3 = \theta_2 \quad (6)$$

and

$$(gk_0)^{-1}\tfrac{1}{2}(R_t)^2 + U(R) = W, \quad (7)$$

where

$$U = -\alpha R^2 - \beta R^3 + \tfrac{1}{4}\gamma R^4, \quad (8)$$

$\gamma = \gamma_1 + 2\gamma_2$, and W= const.

When $\alpha > 0$, the system is linearly unstable. The term $-\beta R^3$ in U accelerates the instability when $R < 0$, and the term $\frac{1}{4}\gamma R^4$ stops the instability when $\gamma > 0$, which occurs when $-0.27 < \beta < 0$. We have two minima of U at

$$R_\pm = (1/2\gamma)[3\beta \pm (9\beta^2 + 8\alpha\gamma)^{1/2}]; \quad (9)$$

$R = R_\pm$ are the static solutions of Eq. (7). By using Eq. (4), we can show that a small disturbance satisfying $i\delta F = \pm(F_{\pm 1} - R_+) = \pm(F_{\pm 3} - R_+) = \mp(F_{\pm 2} - R_+)$ grows when δF is a real quantity. Therefore, $R = R_+$ is an unstable solution. It was confirmed that any small disturbances imposed on $R = R_-$ did not grow. When $\alpha < 0$ and $\gamma > 0$, minima of U appear at $R = 0$ and $R = R_-$. Because these two states are stable, we have a sudden onset of R_- $(= 3\beta/\gamma)$ when α is changed from a negative value to a positive value (see Fig. 1). The equilibrium switches back to $R = 0$ from $R = R_-$ when α is decreased to $-9\beta^2/8\gamma$. This width of hysteresis in α is typically 10^{-2}, so that very careful experimentation is necessary to observe the hysteresis. When the charge density is too large so that $\beta < -0.27$, γ becomes negative, and the R^4 term in U does not stabilize the instability. As a result, the electrons may move to the bottom of the liquid.

Figure 2 shows the equal-height contours of $a(x,y)$ when $R = R_-$ and $\theta_j = 0$. The bottom of the dimples, labeled by A, makes a triangular lattice, and the peaks of the surface, labeled by B, form a hexagonal structure. We have three phases θ_1, θ_2, and θ_3, but two of these are independent because of restriction (6). One can easily find that the introduction of the phases simply causes a parallel translation of the lattice, if θ_j is independent of $\vec{r}$. For instance, if $\theta_2 = 0$ and $\theta_1 = -\theta$, then the whole structure moves by $2\theta_1/k_0\sqrt{3}$ along the direction perpendicular to $\hat{k}_2$. When θ_j is a function of $\vec{r}$, but satisfies $\hat{k}_j\cdot\nabla\theta_j = 0$, Eqs. (7) and (8) are still unchanged. Using $\vec{k}_1 + \vec{k}_3 = \vec{k}_2$ and $\theta_1 + \theta_3 = \theta_2$, we find $\theta_j = \delta k(\hat{k}_j\times\hat{z})\cdot\vec{r}$, which introduces a rotation of the lattice structure through $\delta k/k_0$ ra-

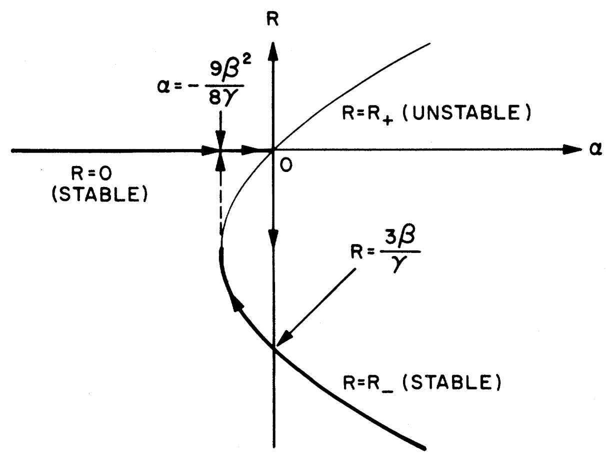

FIG. 1. Amplitude of lattice, R, as a function of α. Thick lines indicate stable solutions.

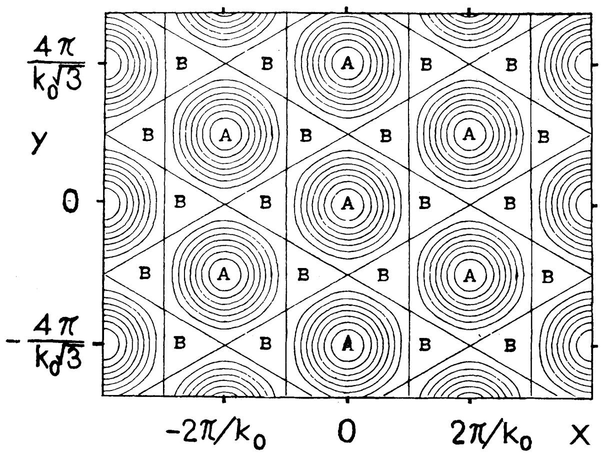

FIG. 2. Equal-height contours of the surface when the lattice is formed. A and B indicate the location of the bottom and the peak, respectively. The contours equally divide the height. $\theta_1 = \theta_2 = \theta_3 = 0$.

dians.

The dimple lattice allows the lattice waves to propagate through it. Let us introduce space-time-dependent phases and discuss the perturbation

$$F_j = R_- \exp[-i\theta_j(\vec{r},t)] \simeq R_-[1 - i\theta_j(\vec{r},t)]. \quad (10)$$

Substituting this expression into (4) and assuming $\theta_j \sim \exp(i\vec{K}\cdot\vec{r} - i\Omega t)$ and $\theta_{-j} = -\theta_j$, we obtain

$$L_1\theta_1 = L_2\theta_2 = L_3\theta_3 = 3\beta R_-(\theta_2 - \theta_3 - \theta_1), \quad (11)$$

where $L_j = (gk_0)^{-1}[-\Omega + v_g^2(\hat{k}_j\cdot\vec{K})^2]$. The condition to obtain the nontrivial solution of (11) yields the dispersion relation

$$L_1L_2L_3 + 3\beta R_-(L_1L_2 + L_2L_3 + L_3L_1) = 0. \quad (12)$$

An interesting case occurs when $\theta_2 = \theta_1 + \theta_3$. The dimple depth is kept constant when this restriction is met. In order to have at least one nonzero θ_j, say θ_1, we need $L_1 = 0$ and one of L_2 and L_3 must be zero; I chose $L_3 = 0$ and $L_2 \neq 0$ without losing generality. We then find from (11) that $\theta_2 = 0$, $\theta_1 = -\theta_3$, and $(\hat{k}_1 \pm \hat{k}_3)\cdot\vec{K} = 0$, so that $\vec{K} \parallel \hat{k}_2$ or $\vec{K} \perp \hat{k}_2$. The relation between the phases, $\theta_2 = 0$ and $\theta_1 + \theta_3 = 0$, indicates that the dimple displacement is perpendicular to $\hat{k}_2$. Therefore, the wave propagating along $\hat{k}_2$ is the shear wave and the one propagating perpendicular to $\hat{k}_2$ is the compressional wave. From $L_1 = 0$, the dispersion relations for the shear and the compressional waves are found to be

$$\Omega^2 = \tfrac{1}{2} v_g^2 |\vec{K}|^2 \text{ and } \Omega^2 = \tfrac{3}{4} v_g^2 |\vec{K}|^2. \quad (13)$$

The symmetry of the lattice constrains the shear waves to propagate in the six directions $\{\pm\hat{k}_1, \pm\hat{k}_2, \pm\hat{k}_3\}$ and the compressional waves to propagate along $\{\pm(\hat{k}_1 + \hat{k}_2), \pm(\hat{k}_2 + \hat{k}_3), \pm(\hat{k}_3 - \hat{k}_1)\}$. The observation of these lattice waves is very easy because the frequency is typically a few hertz.

Although only the liquid-vapor system has been analyzed, our theory applies to the ^{3}He-^{4}He system if $\rho_3 + \rho_4$ and $g(\rho_4 - \rho_3)/(\rho_4 + \rho_3)$ are substituted for ρ and g. In the liquid-vapor system, the experiments have to be done below the λ point; otherwise the boiling of the liquid[6] disturbs the surface.[7]

The author appreciates numerical work done by B. C. Chambers and discussions with D. S. Fisher, C. C. Grimes, A. Hasegawa, P. M. Platzman, and C. M. Varma.

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