[8]V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas* (Pergamon, Oxford, 1970), p. 267.
[9]K. G. Estabrook, E. J. Valeo, and W. L. Kruer, Phys. Fluids 18, 1151 (1975).
[10]J. W. Shearer, Phys. Fluids 14, 183 (1971).
[11]T. W. Johnston and J. M. Dawson, Phys. Fluids 16, 722 (1973).
[12]J. J. Thomson, W. L. Kruer, A. Bruce Langdon, Claire Ellen Max, and W. C. Mead, Phys. Fluids 21, 707 (1978).
[13]Julian Crowell and R. H. Ritchie, J. Opt. Soc. Am. 60, 794 (1970).

Goodness of Ergodic Adiabatic Invariants

Edward Ott
Laboratory of Plasma Studies and Department of Electrical Engineering, Cornell University, Ithaca, New York 14853
(Received 15 March 1979)

For a "slowly" time-dependent Hamiltonian system exhibiting ergodic motion, the volume inside the hypersurface on which the Hamiltonian equals a constant is an adiabatic invariant. It is shown that the error in the constant is diffusive and scales as $(\tau_C/\tau)^{1/2}$, where τ_C is a certain correlation time of the ergodic motion, and τ is the time scale over which the Hamiltonian changes.

The importance of ergodically wandering solutions of Hamilton's equations has been demonstrated in a variety of plasma-physics problems.[1,2] Recently, Lovelace[2] has considered the compression of a field-reversed ion ring. After assuming that the motion of the ring ions should be ergodic in a plane transverse to the toroidal direction, he demonstrated an adiabatic invariant for the ergodically moving ion. Although his derivation was specific to the ion-ring problem, the invariant is actually a very general one. Namely, for a "slowly" time-dependent Hamiltonian system exhibiting ergodic motion in N spatial dimensions $(\vec{q})$, the volume of $2N$-dimensional phase space $(\vec{q},\vec{p})$ within the hypersurface $H(\vec{p},\vec{q},t) = \text{const}$ (where H is the Hamiltonian) is an adiabatic invariant. (Indeed the existence of this adiabatic invariant is already appreciated in statistical mechanics,[3] and the invariant may be associated with the system entropy. In statistical mechanics N is large, whereas $N=2$ is of interest for Refs. 1 and 2.) The generality of the ergodic invariant suggests that it may be useful in a wide variety of other plasma-physics problems where ergodic particle motion is prevalent. Motivated by this, the present work attempts to evaluate the goodness of the ergodic adiabatic invariant. That is, since the adiabatic invariant is only approximately conserved, how good is the approximation? For the case of the familiar $N=1$ adiabatic invariant $(\oint p\,dq)$ of a particle exhibiting rapid almost periodic motion (e.g., the magnetic moment), the average error in assuming that $\oint p\,dq$ is conserved can be exponentially small in τ, where τ is the time scale over which the Hamiltonian changes.[4] In contrast, it is shown here that, for the ergodic invariant, the error is typically proportional to $\tau^{-1/2}$, and it is shown how to calculate the error.

In order to present a brief heuristic demonstration of the ergodic adiabatic invariant, suppose that the existence of three widely separated time scales, $\tau \gg T \gg \tau_w$, where τ_w is the time it takes the system to wander over the surface $H = \text{const}$, where H is the Hamiltonian, and in computing τ_w one uses the orbit obtained from Hamilton's equations with the explicit slow time dependence of H neglected (since τ is large). The exact distribution function of the system is $f = \delta(\vec{p} - \vec{p}(t))\,\delta(\vec{q} - \vec{q}(t))$, where $\vec{p}(t)$ and $\vec{q}(t)$ are solutions of the exact equations of motion. According to the ergodic theorem,

$$\langle f\rangle_T \cong K(t)\delta(H(\vec{p},\vec{q},t) - H_0(t)), \tag{1}$$

where $\langle \cdots \rangle_T$ denotes an average over the time scale T. Note that K and H_0 evolve on the slow time scale τ (as does H). (For $\partial H/\partial t = 0$ the Hamiltonian is a constant of the motion and H_0 is just a constant, but $\partial H/\partial t \neq 0$ is of interest here.) The principal use of the ergodic invariant is that it will determine the time dependence of H_0. If a surface in $(\vec{p},\vec{q})$ phase space is evolved (with each point on the surface following a system orbit), then the volume inside that surface is conserved.[5] Since (1) represents a distribution function, the surface $H = H_0$ evolves in this manner.

Thus we obtain the desired result:

$$\mathcal{J}(H_0(t),t)=\int U(H_0(t)-H(\vec{p},\vec{q},t))\,d^Np\,d^Nq \qquad (2)$$

is approximately invariant [(1) is approximate], where $U(\cdots)$ is the unit step function. [Note that, for $N=1$, Eq. (2) reduces to $\oint p\,dq$.] Thus, if the Hamiltonian at time $t=0$ has a value $H_0(0)$, its value at any subsequent time, $H_0(t)$, may be approximately determined from $\mathcal{J}(H_0(t),t)=\mathcal{J}(H_0(0),0)$.

The problem will now be considered in a more formal way and the error in the adiabatic approximation obtained. To do this consider an ensemble of initial conditions uniform on the hypersurface $H(\vec{p},\vec{q},0)=H_0(0)$, so that there exists an initial distribution function

$$F(\vec{p},\vec{q},0)=K(0)\delta(H(\vec{p},\vec{q},0)-H_0(0)), \qquad (3)$$

where, for convenience, we take $\int F(\vec{p},\vec{q},0)d^Np\,d^Nq\equiv 1$. One then asks how this initial distribution function evolves with time: $F(\vec{p},\vec{q},t)$. This evolution is obtained from the Vlasov equation,

$$\frac{\partial F}{\partial t}+\frac{\partial H}{\partial \vec{p}}\cdot\frac{\partial F}{\partial \vec{q}}-\frac{\partial H}{\partial \vec{q}}\cdot\frac{\partial F}{\partial \vec{p}}=0. \qquad (4)$$

From (4) it is clear that at any subsequent time, F will retain its initial character, viz., it is a δ function on a surface in $(\vec{p},\vec{q})$ space. Furthermore, it follows from the adiabatic invariant (to be demonstrated more formally below) that the singular surface of the exact F will be close to the surface $H=H_0$, with $H_0(t)$ determined from the constancy of J [cf. Eq. (2)]. No attempt will be made to obtain the details of how the singular surface of the exact F deviates from $H=H_0(t)$. Rather, an average of this deviation will be found. To do this, multiply (4) by $(H-H_0)^2$ and integrate over phase space, with the result

$$\frac{d}{dt}\langle(\Delta H)^2\rangle=2\int F\left(\frac{\partial H}{\partial t}-\frac{dH_0}{dt}\right)(H-H_0)d^Np\,d^Nq, \qquad (5)$$

where $\langle(\Delta H)^2\rangle\equiv[\int(H-H_0)^2Fd^Np\,d^Nq]$. The problem then reduces to obtaining a sufficiently accurate approximation for F to insert into the right-hand side of (5). To do this, a multiple-time-scale analysis[6] will be used.

The following expansion is now introduced:

$$H(\vec{p},\vec{q},t)=h(\vec{p},\vec{q},\tau_2), \qquad (6a)$$

$$F=F_0(\vec{p},\vec{q},\tau_2)+\epsilon F_1(\vec{p},\vec{q},\tau_1,\tau_2)+O(\epsilon^2), \qquad (6b)$$

$$\tau_1=t,\quad \tau_2=\epsilon t,\quad \epsilon\ll 1, \qquad (6c)$$

where the small parameter ϵ is introduced to emphasize that H evolves slowly, in some sense.

Inserting (6) in (4) and expanding to orders ϵ^0 and ϵ^1, one obtains the following two equations:

$$\frac{\partial h}{\partial \vec{p}}\cdot\frac{\partial F_0}{\partial \vec{q}}-\frac{\partial h}{\partial \vec{q}}\cdot\frac{\partial F_0}{\partial \vec{p}}=0, \qquad (7)$$

$$\frac{\partial F_1}{\partial \tau_1}+\frac{\partial h}{\partial \vec{p}}\cdot\frac{\partial F_1}{\partial \vec{q}}-\frac{\partial h}{\partial \vec{q}}\cdot\frac{\partial F_1}{\partial \vec{p}}=-\frac{\partial F_0}{\partial \tau_2}. \qquad (8)$$

The solution to (7) subject to (3) is

$$F_0=k(\tau_2)\delta(h(\vec{p},\vec{q},\tau_2)-h_0(\tau_2)) \qquad (9)$$

with $k(0)=K(0)$, $h_0(0)=H_0(0)$, but with $k(\tau_2)$ and $h(\tau_2)$ so far otherwise arbitrary. To determine $k(\tau_2)$ and $h(\tau_2)$ one needs to use Eq. (8). In the spirit of the multiple-time-scale method, $k(\tau_2)$ and $h(\tau_2)$ must be chosen so that F_1 obtained from (8) does not grow secularly on the time scale τ_1. This is necessary in order that the expansion (6b) remain valid for long times $\tau_2\sim O(1)$. Multiplying (8) by $g(h)$, an arbitrary function of h, and integrating over phase space, one obtains

$$(\partial/\partial\tau_1)\int g(h)F_1d^Np\,d^Nq$$
$$=-\int g(h)(\partial F_0/\partial\tau_2)d^Np\,d^Nq.$$

Clearly, in order to avoid secular behavior of $\int g(h)F_1d^Np\,d^Nq$, one must have

$$\int g(h)(\partial F_0/\partial\tau_2)d^Np\,d^Nq=0.$$

Using (9) in this condition on F_0 one obtains

$$g(h_0)\frac{d}{d\tau_2}\int k(\tau_2)\delta(h-h_0)d^Np\,d^Nq$$
$$+\frac{dg(h_0)}{dh_0}k(\tau_2)\frac{d}{d\tau_2}\int U(h-h_0)d^Np\,d^Nq=0.$$

Since g is arbitrary, it follows that

$$(d/d\tau_2)\int k(\tau_2)\delta(h-h_0)d^Np\,d^Nq=0, \qquad (10a)$$

$$(d/d\tau_2)\int U(h-h_0)d^Np\,d^Nq=0. \qquad (10b)$$

The first condition expresses the conservation of the number of particles, while the second condition is the adiabatic invariant. Equations (10a) and (10b) and the initial values k and h_0 are sufficient to determine $k(\tau_2)$ and $h_0(\tau_2)$. Insertion of F_0, thus obtained, makes the right-hand side of Eq. (5) identically zero. Thus to calculate $\langle(\Delta H)^2\rangle$ it is necessary to obtain F_1. With F_0 specified, one can now solve Eq. (8) for F_1 by integrating over the system orbit (on the time scale τ_1), so that

$$F_1=-\int^{\tau_1}\frac{\partial}{\partial\tau_2}F_0[\vec{P}(\tau_1'),\vec{Q}(\tau_1'),\tau_2]d\tau_1', \qquad (11)$$

where the $\vec{P},\vec{Q}$ trajectories in τ_1' are determined by integrating Hamilton's equations backward in time from $\tau_1' = \tau_1$,

$$\vec{p} = \vec{P}(\tau_1), \quad \vec{q} = \vec{Q}(\tau_1), \tag{12a}$$

$$d\vec{P}/d\tau_1' = \partial h(\vec{P},\vec{Q},\tau_2)/\partial\vec{Q}, \tag{12b}$$

$$d\vec{Q}/d\tau_1' = -\partial h(\vec{P},\vec{Q},\tau_2)/\partial\vec{P}. \tag{12c}$$

[Note that in Eqs. (12), τ_1' and τ_2 are both independent variables. Thus (12) is effectively a problem with a time-independent Hamiltonian.] Now inserting (9) and (11) in (5), reverting to the t variable, interchanging the order of the t and ($\vec{p}$, $\vec{q}$) integrations, and making use of the δ-function identity, $x\delta'(x) = -\delta(x)$, one obtains

$$\frac{d}{dt}\langle(\Delta H)^2\rangle = 2\int^t dt'\left[\int d^Np\,d^Nq\,F_0\left(\frac{\partial H(\vec{p},\vec{q},t)}{\partial t} - \frac{dH_0}{dt}\right)\left(\frac{\partial H(\vec{P}(t'),\vec{Q}(t'),t)}{\partial t} - \frac{dH_0}{dt}\right)\right]. \tag{13}$$

Equation (13) can be concisely expressed by introducing the following autocorrelation function

$$C(s,t) = \int d^Np\,d^Nq F_0\left(\frac{\partial H}{\partial t} - \frac{dH_0}{dt}\right) O_s(t)\left(\frac{\partial H_0}{\partial t} - \frac{dH_0}{dt}\right), \tag{14}$$

where $H = H(\vec{p},\vec{q},t)$, and $O_s(t)$ is an operator which translates values of $\vec{p}$ and $\vec{q}$ backward in time by an amount s by following the trajectory of a time-independent system whose Hamiltonian is $H(\vec{p},\vec{q},t)$ with the explicit time dependence in H frozen [i.e., as in Eqs. (12)]. Thus (13) becomes[7]

$$\frac{d}{dt}\langle(\Delta H)^2\rangle = \int_{-\infty}^{+\infty} C(s,t)ds, \tag{15}$$

where $C(s,t) = C(-s,t)$ has been used, and the integration limits have been set equal to infinity on the basis that the correlation time is much shorter than the slow time scale over which H_0 changes, i.e.,

$$\tau \equiv (d\ln H_0/dt)^{-1} \gg [\textstyle\int_{-\infty}^{+\infty} C(s,t)ds][C(0,t)]^{-1} \equiv \tau_c$$

(a basic assumption of the analysis is that $\tau_c \neq 0$). Note that (15) is in the characteristic form for a diffusion process.

Equation (15) gives the evolution of $\langle(\Delta H)^2\rangle$ in terms of a correlation function involving the orbits for a *time-dependent* Hamiltonian system and is my main result. In practice, application of (15) may be nontrivial, since $C(s,t)$ probably must be evaluated numerically[8] for different values of t. Nevertheless, (15) can yield useful estimates of $\langle(\Delta H)^2\rangle$. In particular, (15) gives the important results that, at $t \sim \tau$,

$$\Delta H_{\rm rms} \sim (\tau_c/\tau)^{1/2}, \tag{16}$$

where $\Delta H_{\rm rms} = [\langle(H - H_0)^2\rangle]^{1/2}$. (Note that $\Delta H_{\rm rms} \to 0$ for $\tau_c/\tau \to 0$, thus verifying that J is indeed an adiabatic invariant.) This result, Eq. (16), is in contrast with the analogous result for the more familiar adiabatic invariant of a particle executing rapid, almost periodic motion in one spatial dimension (e.g., the magnetic moment). Namely, for the one-dimensional adiabatic invariant the average error can be exponentially small[4] in τ [error $\sim A\exp(-B\tau)$, where A and B are constants].

To summarize, an equation [Eq. (15)] has been obtained for the quantity $\langle(\Delta H)^2\rangle$, which measures the goodness of the ergodic adiabatic invariant[9] [Eq. (2)], where the equation involves a correlation function [Eq. (14)] evaluated for the orbits of a *time-independent* Hamiltonian. This equation, Eq. (15), indicates that the spreading in H is diffusive and scales as $(\tau_c/\tau)^{1/2}$ [cf. Eq. (16)].

Thanks are due R. V. Lovelace and J. M. Finn for discussion, and A. N. Kaufman and O. Manley for pointing out the relevance of entropy.

This work was supported by the National Science Foundation.

[1]D. W. Faulconer and R. L. Liboff, Phys. Fluids 15, 1831 (1972); M. N. Rosenbluth, Phys. Rev. Lett. 29, 408 (1972); G. R. Smith, Phys. Rev. Lett. 38, 970 (1977); C. F. F. Karney, Phys. Fluids 21, 1584 (1978); G. R. Smith and A. N. Kaufman, Phys. Fluids 21, 2230 (1978); J.-M. Wersinger, E. Ott and J. M. Finn, Phys. Fluids 21, 2263 (1978); A. J. Dragt and J. M. Finn, J. Geophys. Res. 81, 2327 (1976); A. Fukuyama, H. Momota, R. Itatani, and T. Takizuka, Phys. Rev. Lett. 38, 701 (1977).

[2]R. V. Lovelace, Phys. Fluids 22, 542 (1979).

[3]R. Kubo, *Statistical Mechanics* (North-Holland, Amsterdam, 1965), p. 14.

[4]See, for example, the review by B. V. Chirikov, Fiz. Plazmy 4, 521 (1978) [Sov. J. Plasma Phys. 4, 289 (1978)]. Another interesting reference is that of Lenard, whose approach is similar to ours. Lenard shows that if for $t < 0$ and $t > t_1$, H is constant, then the values of $\oint p\,dq$ at $t < 0$ and $t > t_1$ differ from each other by a quantity which is $O(\epsilon^m)$, however large m may be [A. Lenard, Ann.

Phys. (N.Y.) 6, 261 (1959)].

[5]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1959), p. 250.

[6]For example, R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972), Sect. 1.3.

[7]A calculation similar to that given for $\langle(\Delta H)^2\rangle$ also yields

$$d\langle\Delta H\rangle/dt = \tfrac{1}{2}(\partial/\partial H_0)\int_{-\infty}^{+\infty}C(s,t)ds,$$

which does not affect my estimate, Eq. (16), since $\tau \gg \tau_C$ [where $\langle\Delta H\rangle \equiv \int(H-H_0)F\,d^Np\,d^Nq$]. This result is similar to the relation between friction and diffusion coefficients in weak-turbulence plasma theory [for example, W. M. Manheimer and T. H. Dupree, Phys. Fluids 11, 2709 (1968)].

[8]Correlation functions of particles in ergodic motion have been calculated in several existing works. See, for example, J. M. Finn (to be published), who investigates the ergodicity of ions in an ion-ring equilibrium and utilizes the calculated correlation functions in an analysis of the ring stability.

[9]Another relevant reference, which has recently come to our attention, analyzes the ergodic invariant for a model of Surmac (surface magnetic confinement) particle containment in which particles bounce freely between confining walls which reflect the particles into random directions [Y. C. Lee, T. K. Samec, and B. D. Freid, Phys. Fluids 20, 815 (1977)]. Our analysis does not apply to this model, and, indeed, Lee *et al.* find a diffusion which is smaller than $O(\tau^{-1/2})$.