Alfvén Resonance Effects on Magnetosonic Modes in Large Tokamaks

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The theory of Alfvén resonance effects on the wave modes of a tokamak is extended beyond the incompressible magnetohydrodynamic model to include finite-(ω/Ω_i) effects and compressibility. The discrete spectrum of compressional Alfvén waves consists of a sequence of frequencies with finite damping decrements resulting from the Alfvén resonance. The finite-frequency effects can cause the damping to almost vanish. This permits Alfvén resonance heating via high-Q eigenmodes in large tokamaks.

Tokamaks of the size of the Princeton Large Torus tokamak (PLT)[1] are the first to nominally permit propagation of magnetosonic (i.e., compressional Alfvén) modes at frequencies that are a modest fraction of the ion cyclotron frequency (typically $\omega/\Omega_i \simeq 0.5$). As such, forthcoming wave propagation measurements on these tokamaks will represent the first opportunity to study experimentally the discrete spectrum of wave modes in a tokamak below the ion cyclotron frequency.

In this frequency range, magnetosonic waves may be excited. The antenna determines the parallel and azimuthal components of the wave vector, while the third (i.e., radial) component is determined by the dispersion relation or, more accurately, by the wave propagation equation. Depending on the local value of the Alfvén speed, v_A, this gives evanescent behavior (in the low-density edge of the plasma) or propagating waves where the density is sufficiently high. The transition between the two regions is complicated by the presence of the shear Alfvén resonance. In cold-plasma theory a singularity ($k_r \to \infty$) appears at the point where $\omega \simeq k_\parallel v_A$. (Recall that both ω and $k_\parallel$ are determined by the antenna.) This changes the magnetosonic cutoff into a cutoff-resonance-cutoff triplet. The resonance can lead to energy absorption. The goal of this work is to evaluate how much absorption the shear Alfvén resonance causes. We accomplish this by calculating the reflection coefficient for magnetosonic waves incident on the cutoff-resonance-cutoff triplet.

Previous analyses of the problem[2-4] treated the magnetohydrodynamic (MHD) limit in which $\omega/\Omega_i \to 0$, and, by and large, employed incompressible models. The recent work of Ott, Wersinger, and Bonoli[3] does recognize that the shear Alfvén resonance can damp magnetosonic modes but does not include finite-frequency effects, and specializes its results to the $m=0$ ($k_y=0$) case. Discussions of the MHD spectra[5,6] of a diffuse linear pinch do not treat quantitatively the discrete spectrum. Finite-frequency effects have been considered by Conn and Tataronis,[7] via a Hall term in Ohm's law, but their incompressible model ignores coupling to magnetosonic waves.

Since the dissipation depends on the fields close to the region of the cutoffs and resonance, which typically occupy only a small fraction of the minor radius, we may accurately treat the problem by a slab-geometry model in which x is parallel to the density gradient and z is parallel to the constant magnetic field $\vec{B}_0$. A linear variation of density with x is employed. The neglect of the shear of the magnetic field is justifiable as long as $n \gg m/q$ (n and m are the toroidal and poloidal mode numbers and q is the safety factor). The electrons and ions are treated as separate cold fluids. All field quantities vary as $f(x)\exp(ik_y y + ik_z z - i\omega t)$. The parallel electrical field is neglected (it is shorted out by the high parallel electron conductivity). The remaining components of the electric field are described by

$$(A - k_y^2)E_x + i(D - k_y d/dx)E_y = 0, \tag{1}$$

$$-i(D + k_y d/dx)E_x + (A + d^2/dx^2)E_y = 0, \tag{2}$$

where $A = (\omega^2/v_A^2)[\Omega_i^2/(\Omega_i^2 - \omega^2)] - k_z^2$, $D = (A + k_z^2)\omega/\Omega_i$, and $v_A = c\Omega_i/\Omega_{pi}$. Making the substitutions $E = E_y$ and

$$\psi = (A - k_y^2)^{-1}(A d/dx - k_y D)E_y, \tag{3}$$

we may recast (1) and (2) as a coupled set of first-order differential equations,

$$\frac{d}{dx}E - \frac{k_y D}{A}E = \frac{A - k_y^2}{A}\psi, \tag{4}$$

$$\frac{d}{dx}\psi + \frac{k_y D}{A}\psi = -\left(\frac{A^2 - D^2}{A}\right)E. \tag{5}$$

The right-hand side of Eq. (5) is the cutoff-resonance-cutoff triplet.

The nature of the wave propagation can be seen in the WKB limit by letting d/dx become ik_x in Eqs. (4) and (5). This yields the dispersion relation

$$k_x{}^2 = -k_y{}^2 + (A-D)(A+D)/A\,. \tag{6}$$

Figure 1 graphs this dispersion relation as a function of density and clearly shows the cutoff-resonance-cutoff phenomenon discussed previously. Propagation region I, which vanishes as $\omega/\Omega_i \to 0$, corresponds to the electromagnetic ion-cyclotron wave discussed by Stix[8] and used in ion-cyclotron heating experiments on the C stellarator.[9] The resonance at $A=0$ is called either the shear Alfvén resonance or the perpendicular ion-cyclotron resonance.[8] Region II of Fig. 1 indicates where the compressional magnetosonic mode propagates.

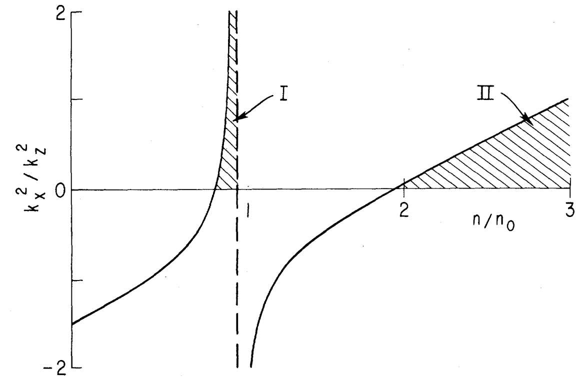

FIG. 1. Representative plot of the normalized wave number k_x/k_z vs density from the dispersion relation (6). Here $\omega = \Omega_i/3$ and $k_y = k_z/2$. The reference density, n_0, is determined by $A=0$.

We put Eqs. (4) and (5) into dimensionless form by defining a scale length $l = [A'(x_0)]^{-1/3}$ and a new coordinate $\xi = (x-x_0)/l$, where $A' = dA/dx$ and x_0 is the point at which $A=0$ (the finite-frequency generalization of the shear Alfvén resonance). For small ξ we retain only the leading-order terms in the series expansions, for A and D. Thus, $A \simeq \xi/l^2$ and $D \simeq S/l^2$, where $S = k_z{}^2 l^2(\omega/\Omega_i)$. Equations (4) and (5) then become

$$\frac{d}{d\xi}E - \frac{MS}{\xi}E - \frac{\xi - M^2}{\xi}\Psi = 0, \tag{7}$$

$$\frac{d}{d\xi}\Psi + \frac{MS}{\xi}\Psi + \frac{\xi^2 - S^2}{\xi}E = 0, \tag{8}$$

where $M = k_y l$ and $\Psi = l\psi$. The equation for E reads

$$\frac{d^2}{d\xi^2}E - \frac{M^2}{\xi(\xi - M^2)}\frac{d}{d\xi}E + \left(\frac{\xi^2 - S^2}{\xi} - M^2 + \frac{MS}{\xi(\xi - M^2)}\right)E = 0. \tag{9}$$

We note the following points about Eq. (9). The equation is singular at $\xi = 0$, the shear Alfvén resonance. The point $\xi = M^2$ is only an apparent singularity[10] as may be seen from Eqs. (7) and (8). The physical significance of the apparent singularity is simply a statement of wave polarization: $\xi\, dE/d\xi = MSE$ at $\xi = M^2$. In the limit $|\xi| \to \infty$ the equation becomes the Airy equation, $d^2E/d\xi^2 + (\xi - M^2)E = 0$, which physically describes a propagating magnetosonic wave for $\xi > M^2$ and an evanescent mode for $\xi < M^2$.

To determine the reflection coefficient, we solve Eq. (9) with the boundary condition that as $\xi \to -\infty$, $E \propto \mathrm{Ai}(M^2 - \xi)$, the evanescent Airy function. We ignore the small portion of the Bi function which should be added since the boundary of the plasma where $E = 0$ is at the finite distance $\xi = -x_0/l$. By integrating Eq. (9) through the Alfvén resonance layer, we find what combination of the Ai and Bi functions we have at $\xi \to \infty$. (Treating the frequency as a Laplace transform variable, we find that causality requires us to go above the singularity at $\xi = 0$ in the complex plane.) We decompose the Ai and Bi functions into a wave incident on the layer $\xi = M^2$ and a reflected wave. We define an amplitude reflection coefficient, R, and from it the fraction, q, of incident power dissipated upon reflection $q = 1 - |R|^2$.

In cases of interest, S and M are small. An analytical estimate of q is then possible. We obtain this by finding series solutions to Eq. (9) about the singular point and by matching these onto the Airy functions at large ξ. In the neighborhood of the singularities, the solutions are (to lowest orders in M and S)

$$E_1 = 1 + [(S/M) + S^2]\xi + O(S^4, \xi^3, S^2\xi^2), \tag{10}$$

$$E_2 = 1 + MS(1 - MS)E_1 \ln\xi + O(S^4, \xi^3, S^2\xi^2). \tag{11}$$

[We assume $M = O(S)$.] The appropriate combination of E_1 and E_2 that matches onto $\mathrm{Ai}(-\xi)$ in the region $M^2 \ll -\xi \sim \xi_0 \ll 1$ is

$$E = [c_1 - (M/S)c_2]E_2 + [(M/S)c_2 + \ln(-\xi_0)(M^2c_2 - MSc_1)]E_1, \tag{12}$$

where $c_1 = \mathrm{Ai}(0) = 0.355$ and $c_2 = -\mathrm{Ai}'(0) = 0.259$. We write $\ln(-\xi_0) = \ln|\xi_0| + i\pi$, and then ignore the $\ln|\xi_0|$ term since it is smaller than the other real terms. For $M^2 \ll \xi \ll 1$ we then have

$$E = \mathrm{Ai}(-\xi) + i(\pi/2\sqrt{3})[M(c_2/c_1)^{1/2} - S(c_1/c_2)^{1/2}]^2 \mathrm{Bi}(-\xi), \tag{13}$$

from which we determine the fractional power absorbed,

$$q = (2\pi/\sqrt{3})[M(c_2/c_1)^{1/2} - S(c_1/c_2)^{1/2}]^2. \tag{14}$$

The MHD limit is given by $S \to 0$. We see that the effect of the finite frequency is either to increase or to decrease q depending on the sign of M. If $S = Mc_2/c_1$, we have $q = 0$.

In particular, if $S = M = 0$, we have $q = 0$ and, hence, there is no energy absorption. Indeed in this limit Eq. (9) reduces to the Airy equation. In the same limit, i.e., $\omega/\Omega_i \to 0$ and $k_y \to 0$, the equation obtained assuming an incompressible equation of state[11] is singular and predicts absorption. In low-β plasmas such as tokamaks, our model is the more appropriate because there is no force linear in the wave amplitude which will produce a plasma acceleration parallel to the equilibrium magnetic field. (Such a force is assumed in incompressible models.)

In Fig. 2 we compare Eq. (14) with the value of q obtained by numerically integrating Eqs. (7) and (8). We see that there is good agreement for $|S|$, $|M| \lesssim 0.5$. In particular the line along which $q = 0$ is accurately given by Eq. (14).

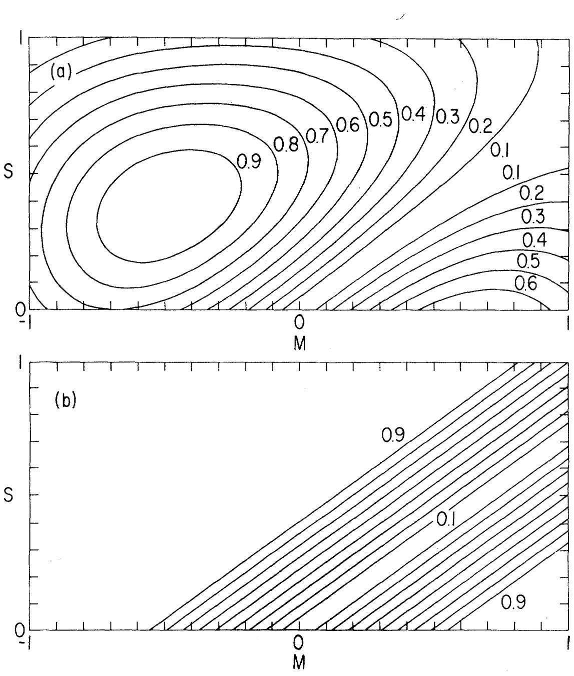

FIG. 2. Contours of constant $\sqrt{q}$. (a) Results obtained by numerically integrating (7) and (8). (b) Analytic result (14).

In a cyclindrical version of (4) and (5), which is an appropriate model for a tokamak, we must impose a regularity boundary condition on the cylindrical axis in addition to the condition that $E = 0$ at the surface. The equations then become a complex-eigenvalue problem and integration along an appropriate complex contour generates a discrete set of oscillating, damped eigenmodes. Numerical integrations find the expected eigenvalues which have a minimum in the damping decrement where q is small. Figure 3 gives typical results.

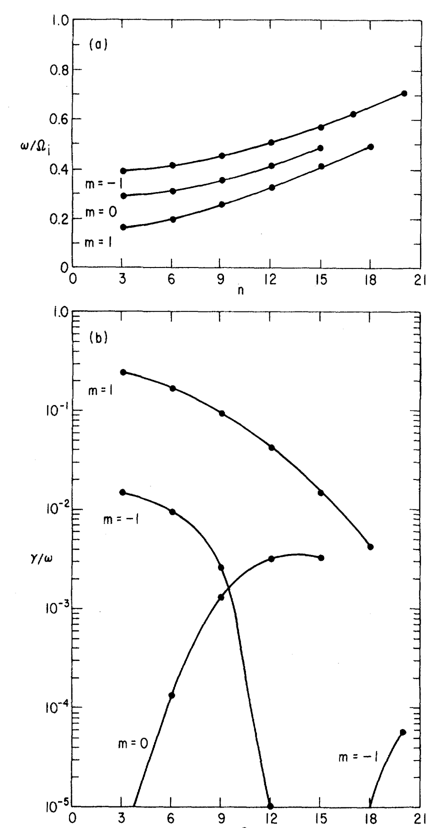

FIG. 3. Representative points of the discrete spectrum of a diffuse cylindrical plasma column. The frequency ω and the damping decrement are plotted for a discrete set of parallel wave numbers $k_\parallel = n/R$. Lines serve simply to identify the m number. Computations assumed a parabolic density profile with $\omega_{pi,0}^2 a^2/c^2 = 3 \times 10^2$ and $R/a = 3$ characteristic of the PLT at a central density of 10^{14} protons/cm^3.

To illustrate our result let us compute q for the PLT taking $k_y = -m/a$, $k_z = n/R$, $a = 40$ cm, $R = 1.3m$, and assuming a parabolic density profile. We find

$$q = 0.1m^2\left(\frac{B_{40}}{a_{40}f_{25}}\right)^{4/3}\left(\frac{1}{n_{14}}\right)^{2/3}\left[1+\left(\frac{a}{R}\right)^2\frac{n^2}{10m}\left(\frac{f_{25}}{B_{40}n_{14}a_{40}^{\,2}}\right)^{1/3}\right]^2, \tag{15}$$

where B_{40} is the toroidal field in units of 40 kG, f_{25} is the frequency of the wave generator in units of 25 MHz, n_{14} is the central density in units of 10^{14} cm^{-3}, and a_{40} the minor radius in units of 40 cm. Thus, if $m = -1$, q is small for $n \simeq \pm 10$. With this value of n the wave energy propagates away from the antenna reasonably quickly (since $k_\perp \sim k_z$) so that the parallel damping length is long. We expect, therefore, that a high-Q eigenmode will be excited. Practically, this allows us to combine the high antenna loading associated with toroidal eigenmodes[12] with the Alfvén wave dissipation process.

The physics of the dissipation mechanism has been discussed previously. Finite-temperature and parallel-electric-field effects[13] change Eqs. (1) and (2) into a higher-order system without singularities. The dissipated energy is linearly mode converted[14] into a "kinetic-Alfvén" wave, which damps rapidly via linear electron Landau damping. Hence, the energy absorbed by the shear Alfvén resonance shows up as electron heating near the resonance.

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