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Aharonov-Bohm Effect from the "Hydrodynamical" Viewpoint

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It is shown that the adoption of the hydrodynamical viewpoint in quantum mechanics provides a way to explain experimental results while giving electromagnetic potentials no more physical significance than they have in classical physics.

In the "hydrodynamical" formulation of quantum mechanics,[1] the Schrödinger equation is replaced by a set of nonlinear equations for the density of probability ρ and the density current $\vec{J}$. This formulation entered the controversial history of the Aharonov-Bohm effect[2] because fields, and not potentials, appear in the hydrodynamical equations so that, in the time evolution of ρ and $\vec{J}$ described by these equations, there is no room for potential effects, in striking contrast with the theory of Aharonov-Bohm based upon the Schrödinger equation.[3] In the theoretical prediction of the Aharonov-Bohm effect one deals with electrons moving in a multiply connected configuration space. The purpose of this paper is to point out that the relationship between the Schrödinger equation and hydrodynamical equations undergoes a significant modification when one considers multiply connected domains and that no paradox occurs once the correct relationship between the two formulations is taken into account. Let us briefly review some facts about quantum mechanics in multiply connected regions such as the exterior of an impenetrable solenoid. In this region there is a class of transformations of the electromagnetic potentials which are permissible, in the sense that they leave unchanged both the behavior at infinity and the fields in the allowed domain, and which are not eliminated by the usual gauge $\operatorname{div}\vec{A}=0$. Let one such transform carry $\vec{A}$ into $\vec{A}'$: The Schrödinger equation is not invariant under this transform unless $(e/c)\oint_\gamma(\vec{A}-\vec{A}')\cdot d\vec{s}=nh$, the contour embracing once the forbidden region; in this case to $\vec{A}$ and $\vec{A}'$ correspond unitarily equivalent Hamiltonians. Thus, for a given field, one obtains as many nonequivalent Hamiltonians. Thus, for a given field, one obtains as many nonequivalent Hamiltonians as there are points in the unit circle. There are a number of different ways to attain this result. For instance, it has been shown within the framework of the path-integral formalism that there are as many distinct propagators as there are distinct unitary representations of the homotopy group.[4] If one regards the ideal multiply connected situation as the limit of a simply connected one—e.g., when a potential barrier is raised around the solenoid—one is led to conclude that the appropriate Hamiltonian is obtained when $\vec{A}$ is the Stokesian vector potential of the magnetic field inside the cylinder: This is the Aharonov-Bohm effect *in nuce*.

We will now show that the hydrodynamical approach to the question assigns a definite role to all these Hamiltonians, without calling for hidden magnetic fields. Consider the following set of equations:

$$\begin{aligned}\partial\vec{u}/\partial t &= -(\hbar/2m)\mathrm{grad}(\mathrm{div}\,\vec{v}) - \mathrm{grad}(\vec{v}\cdot\vec{u}),\\ \partial\vec{v}/\partial t &= (1/m)\vec{F} - (\vec{v}\cdot\mathrm{grad})\vec{v} + (\vec{u}\cdot\mathrm{grad})\vec{u} + (\hbar/2m)\Delta\vec{u},\end{aligned} \tag{1}$$

where

$$\vec{F} = e[\vec{E} + c^{-1}\vec{v}\times\vec{H}].$$

The problem of finding the solutions of Eqs. (1), under the additional conditions that $\vec{u}$ be a gradient and that

$$\mathrm{rot}\,\vec{v} + (e/mc)\vec{H} = 0, \tag{2}$$

is shown to be formally equivalent to the solution of the Schrödinger equation for a particle moving in the electromagnetic field $\vec{E}, \vec{H}$ with

$$\begin{aligned}&\psi = \exp(R + iS),\\ &\mathrm{grad}R = (m/\hbar)\vec{u},\\ &\mathrm{grad}S = (m/\hbar)[\vec{v} + (e/mc)\vec{A}];\ \mathrm{rot}\vec{A} = \vec{H}.\end{aligned} \tag{3}$$

Equations (1) were derived by Nelson[5] in his well-known attempt to give a stochastic interpretation of quantum mechanics. Since we shall use Eqs. (1) as our starting point, our remarks about the Aharonov-Bohm effect may also be considered as being made from the standpoint of stochastic mechanics.

Equations (1) are related to the usual hydrodynamical formulation through

$$\vec{u} = (\hbar/2m)(1/\rho)\mathrm{grad}\rho, \quad \vec{v} = \vec{J}/\rho. \tag{4}$$

Consider now the initial-value problem (1) and (2) in the exterior domain of a reflecting cylinder and assume that $\vec{E} = \vec{H} = 0$ in this domain. If we let $\vec{w} = \vec{v} + i\vec{u}$, Eqs. (1) and (2) assume the following form reminiscent of ordinary hydrodynamics:

$$(\partial\vec{w}/\partial t) + (\vec{w}\cdot\mathrm{grad})\vec{w} + (i\hbar/2m)\Delta\vec{w} = 0, \tag{5}$$

$$\mathrm{rot}\ \vec{w} = 0. \tag{5'}$$

For our present purposes it will not be necessary to fulfill the delicate task of specifying appropriate boundary conditions on the surface of the cylinder beyond the natural requirement that $v_n = 0$ there. Equation (5′) and Clebsch's theorem now tell us that:

$$\vec{w} = \mathrm{grad}\varphi + \vec{a}, \tag{6}$$

where $\mathrm{rot}\,\vec{a} = 0$, $\mathrm{div}\,\vec{a} = 0$. Since $\vec{u}$ is a gradient $\vec{a}$ is a real vector; it satisfies the equation

$$\oint_\gamma \vec{a}\cdot d\vec{s} = \oint_\gamma \vec{w}(t)\cdot d\vec{s}, \tag{7}$$

where the contour γ winds once around the cylinder. The right-hand side of (7) is independent of t. To see this, choose a γ lying on the surface of the cylinder, and let γ be carried into γ' at time t' by the flow $\vec{v}$: γ', too, lies on the cylinder. Then $\oint_{\gamma'}\vec{w}(t')\cdot d\vec{s} = \oint_\gamma \vec{w}(t)\cdot d\vec{s}$ follows from (5) much as in usual hydrodynamics. Moreover, $\oint_{\gamma'}\vec{w}(t')\cdot d\vec{s} = \oint_\gamma \vec{w}(t')\cdot d\vec{s}$ thanks to Eq. (5′). Then, once $\vec{w}(0)$ is given, Eq. (6) can be satisfied at all times by the choice

$$\vec{a} = (1/2\pi r)[\oint_\gamma \vec{w}(0)\cdot d\vec{s}]\vec{\theta},$$

where $\vec{\theta}$, at a given point, is the unit vector normal to the plane through the point and the axis of the cylinder.

Writing $\psi = \exp[i(m/\hbar)\varphi^*]$, $\vec{A} = -(mc/e)\vec{a}$, one easily deduces from Eqs. (5) and (5′)

$$\frac{\partial\psi}{\partial t} = -\frac{i}{2m\hbar}\left(-i\hbar\,\mathrm{grad} - \frac{e}{c}\vec{A}\right)^2\psi. \tag{8}$$

This is the Schrödinger equation for a particle moving outside the cylinder; indeed, it is exactly the equation that one would write if he were willing to take into account a magnetic field inside the cylinder with $\vec{A}$ as its Stokesian vector potential.[6] Of course this magnetic field is completely fictitious. The point is that the (formal) equivalence between the Schrödinger equation and Eqs. (1) and (2) is lost when multiply connected regions are taken into account. The evolution of $\vec{w}$ (or $\rho, \vec{J}$) cannot be found by prescribing one Schrödinger equation once and forever. Because of the multiple connectedness of the domain ψ does not completely describe the hydrodynamical (irrotational) field $\vec{w}$. An additional datum must be supplied, namely, the time-preserved quantity $\oint_\gamma \vec{w}$

$\cdot d\vec{s}$. As a result, the time evolution in the space of (irrotational) data $\vec{w}$ does not project on a uniquely defined evolution in the space of data ψ. Suppose now that one is interested in the time evolution of a beam of particles initially directed straight towards the cylinder. Since $\oint \vec{v} \cdot d\vec{s} = 0$, the appropriate Schrödinger equation is the one with $\vec{A} = 0$. This means that things go in exactly the same way, no matter what fields there are inside the cylinder. However, using a different Hamiltonian to let the same initial situation evolve is inconsistent with Eqs. (1) and (2).

At this point one might be tempted to say that some evidence of the effectiveness of the potential can still be supplied while keeping away from idealizations such as multiply connected domains. To this end one should screen the solenoid by means of a potential barrier of increasing height and then compare the time evolution of a given initial situation with the current in the solenoid switched on or off. Now, if one observes the evolution of the given wave packet in the two cases, he will find that these evolutions are different and that the difference shows no tendency to vanish when the barrier is increased. This is no evidence at all of an effect of the potential because the same initial wave function ψ gives rise to two different initial conditions in the space of data[7] ρ and $\vec{J}$ whose evolution, on the other hand, being governed by Eqs. (1), depends only on field strengths and not on potentials. One should instead compare the time evolutions subsequent to the same initial assignment of $\vec{w}$, but this cannot be achieved because of the condition (2) which forbids that a field $\vec{v}$ which can be deduced by some wave function in the presence of nonzero unscreened magnetic field can also be deduced by a wave function with zero magnetic field. Thus one of the terms of comparison is actually lacking.

Now, let Φ denote the magnetic flux inside the solenoid. When the potential barrier is increased, we approach the ideal multiply connected situation in which a beam with $\oint \vec{v} \cdot d\vec{s} = -e\Phi/mc$ (and not one with $\oint \vec{v} \cdot d\vec{s} = 0$) is scattered by an impenetrable cylinder. The appropriate Hamiltonian is now, according to our view, the one where $\vec{A}$ is the vector potential of the magnetic field inside the hardened cylinder. Thus the hydrodynamical theory of the ideal case can indeed be used to approximate real cases.

In conclusion our analysis, though rather heuristic in character, shows that the adoption of the hydrodynamical viewpoint in quantum mechanics provides a way to explain experimental results while giving electromagnetic potentials no more physical significance than they have in classical physics.

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