which it is impossible to satisfy all the bonds in the ground state. Surprisingly, Betts once again finds that the ground state belongs to minimum $|M_z|$! However, the ground state is now highly degenerate, in contrast to the previous case where Frobenius's theorem applies. Similarly, our reference Hamiltonian (7) will, after change of sign, have not only a unique (or doublet) ground state belonging to $I=0$ and $m=0$ (or $I=\frac{1}{2}$, $m=\pm\frac{1}{2}$) but a dense spectrum of low-lying states, *almost* degenerate with the ground states (in the limit $N \to \infty$, $I_{\max} \propto N$).

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## Critical Correlation Function and Exponent $\eta$: A Sum Rule

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A sum rule is derived from the connection of the critical specific heat with the critical correlation function in the disordered symmetric state. The "Ornstein-Zernike hole" which appears above the critical point must, by virtue of this sum rule, be canceled exactly by the "Fisher-Langer tail." It follows that the anomalous dimension index $\eta$ is fixed by the shape of the spectral function. A spectral function satisfying certain general properties yields $\eta \approx 0.04$.

The order-parameter–order-parameter correlation function, $G(r,\kappa)$, and its Fourier transform, $g(k^2,\kappa^2)$ describe correlation in configuration and wave-number space, respectively. The critical-region temperature dependence enters through the inverse correlation length $\kappa$ and is expressed by the differences $\Delta G(r,\kappa) \equiv G(r,\kappa) - G(r,0)$ and $\Delta g(k^2,\kappa^2) \equiv g(k^2,\kappa^2) - g(k^2,0)$. In many cases a sufficiently accurate approximation is the Ornstein-Zernike[1] formula

$$g_{OZ}(k^2,\kappa^2) = C_{OZ}/(k^2+\kappa^2), \tag{1}$$

where $C_{OZ}$ is a positive constant. In this approximation the critical variation is given by the negative-definite expression

$$\Delta g_{OZ} = -\frac{C_{OZ}\kappa^2}{k^2(k^2+\kappa^2)}. \tag{2}$$

But with the experimental precision which can presently be achieved,[2] a more accurate representation of $g(k^2,\kappa^2)$ is needed. This is also true in theoretical calculations where $g(k^2,\kappa^2)$ enters as an essential ingredient. An accurate form for $g(k^2,\kappa^2)$ is especially important in calculations where fluctuations of large $k$ enter in an essential way, such as in studies of the critical viscosity[3] and of the critical attenuation of sound. For $k \gg \kappa$ the correlation function must asymptotically approach its $\kappa=0$ Green[4]-Fisher[5] form

$$g(k^2,0) = C_{GF}k^{-2+\eta} \tag{3}$$

assuming that $k$ is still small enough to be in the critical region. Here $C_{GF}$ is a constant and $\eta$ is the anomalous dimension index. $\eta$ is the most basic of the critical indices and its calculation is one of the central problems in the theory of phase transitions. A related problem, generally studied independently is the detailed $k^2$ and $\kappa^2$ dependence of $g(k^2,\kappa^2)$, i.e., the "shape" of the function. In this Letter we discuss a sum rule which is equivalent to a certain identity in Lagrangian field theory.[6] This sum rule imposes an intimate con-

 

nection between $\eta$ and the shape of $g(k^2, \kappa^2)$. Once the shape of $g(k^2, \kappa^2)$ is established, the value of $\eta$ is fixed by this sum rule.

The sum rule follows from the critical-region temperature dependence of the internal energy of the system. For a short-range pair-interaction potential the critical energy is proportional to $\Delta G(r, \kappa)$ where the force range is characterized by some small value $r \ll \kappa^{-1}$. We therefore set $r = 0$ and substitute from Eq. (2) to find $\Delta G_{OZ}(0, \kappa) = -\kappa/4\pi$. (We mainly treat the three-dimensional case.) This would require that $\nu$, the critical index for the correlation length, be also the energy critical index. Temperature differentiation would then yield the specific heat, *but with the wrong critical index.* The exponent would be larger than the correct exponent $\alpha$ by $2\nu - 1$. (Here we have used the scaling law $3\nu = 2 - \alpha$.) As $\nu$ is generally larger than $\frac{1}{2}$, its mean-field value, it follows that the negative definiteness of Eq. (2) is qualitatively inaccurate. The correct correlation function $\Delta g$ is known, in fact, to have a positive tail of the Fisher-Langer[7] type in the $k \gg \kappa$ range. We therefore assert that this "Fisher-Langer tail" must exactly cancel the "Ornstein-Zernike hole." This requirement is expressed by the sum rule

$$\Delta G(0, \kappa) = (1/2\pi^2)\int_0^\infty dk\, k^2\, \Delta g(k^2, \kappa^2) = 0. \tag{4}$$

At this point the reader may well object that we have thrown the baby out with the washwater. We hasten to explain that Eq. (4) refers only to the scaling portion of $\Delta G$ and does *not* imply the complete absence of any critical specific heat. More precisely, we should write Eq. (4) as $\lim_{\kappa \to 0} \kappa^{-1+\eta} \Delta G = 0$. The true critical behavior ensues from the high-$k$ deviation from scaling, such as the Debye cutoff on the Fisher-Langer tail.[8]

We have verified the two-dimensional version of Eq. (4) by explicit numerical integration using Tracy and McCoy's[9] tabulation for the two-dimensional Ising model. Similarly, we have confirmed in the screening approximation[10,11] or $n^{-1}$ expansion[12,13] that the Fisher-Langer tail does indeed have the strength necessary to cancel the Ornstein-Zernike hole exactly. The general case is most conveniently handled by means of the spectral function $\mathrm{Im}\Delta g(-|k^2|, \kappa^2)$. This is the analytic continuation of $\Delta g$ in the complex $k^2$ plane to the upper side of the cut running along the negative real axis. With the basic assumption of no singularities except along the cut, Cauchy's theorem gives

$$\Delta g(k^2, \kappa^2) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{dz}{z - k^2}\, \mathrm{Im}\Delta g(z, \kappa^2). \tag{5}$$

We note first that, for $k^2 \gg \kappa^2$, $z$ can be dropped from the denominator, which can then be taken outside the integral. This would have $\Delta g(k^2, \kappa^2)$ varying as $k^{-2}$, which would be wrong. Its actual $k$ dependence is more rapid. Therefore

$$\int_{-\infty}^{0} dz\, \mathrm{Im}\Delta g(z, \kappa^2) = 0, \tag{6}$$

which fixes the strength of the $\delta$ function at $z = -\kappa^2$ in terms of the continuous variation of $\mathrm{Im}\Delta g$ for $z \neq -\kappa^2$.

Substituting Eq. (5) into Eq. (4) and interchanging the order of integration bring in

$$\frac{1}{2\pi^2}\int dk \frac{k^2}{-k^2 - |z|} = -\frac{1}{2\pi^2}\int dk + \frac{|z|}{2\pi^2}\int_0^\infty \frac{dk}{k^2 + |z|}$$

$$= -\frac{1}{2\pi^2}\int dk + \frac{1}{4\pi}|z|^{1/2}. \tag{7}$$

The first term of Eq. (7) is a constant which disappears in the subsequent $z$ integration, by virtue of Eq. (6). The remaining term puts Eq. (4) into the form

$$\Delta G(0, \kappa) = (1/4\pi^2)\int_{-\infty}^{0} dz |z|^{1/2} \mathrm{Im}\Delta g$$

$$= (1/4\pi^2)\int_{-\infty}^{0} dz (|z|^{1/2} - \kappa)\, \mathrm{Im}\Delta g = 0. \tag{8}$$

The second line of Eq. (8), obtained by subtracting $(-\kappa/4\pi^2)$ times Eq. (6), is the more convenient version because the pole no longer contributes explicitly.

In order to proceed further we introduce the threshold at $|z|^{1/2} = l\kappa$, the scaling variable $u = |z|^{1/2}/l\kappa$, and the threshold function $F(u)$. $F(u)$ vanishes for $0 \leq u < 1$ and approaches unity asymptotically as $u \to \infty$. By analytic continuation of Eq. (3) we find for $0 \leq u < 1$,

$$\mathrm{Im}\Delta g = -\mathrm{Im} g(k^2, 0) = -\mathrm{Im} C_{GF}(i|k|)^{-2+\eta}$$

$$= C_{GF} \sin(\tfrac{1}{2}\pi\eta)|k|^{-2+\eta}$$

$$= C_{GF} \sin(\tfrac{1}{2}\pi\eta)\, l^{-2+\eta} \kappa^{-2+\eta} u^{-2+\eta}.$$

For $u \geq 1$ this gets multiplied by $1 - F(u)$. Dropping powers of $\kappa$ and constant factors, we obtain the scaled version of Eq. (8) as

$$\int_0^\infty du\, u^\eta (1 - 1/lu)[1 - F(u)] = 0. \tag{9}$$

The range below threshold yields $(1+\eta)^{-1} - (l\eta)^{-1}$. Thus the linearized version of the sum rule, valid for $\eta \ll 1$, is

$$\eta^{-1} = l + \int_1^\infty du (l - u^{-1})[1 - F(u)]. \tag{10}$$

The Fisher-Burford[14] approximant corresponds essentially to a step function for $F(u)$ with threshold at $l=(0.15)^{-1}$. The integral makes a negligible contribution, so that Eq. (10) requires the relatively large value $\eta=0.15$.

A more realistic threshold function has been proposed[15,16] based on extending the Fisher-Langer tail all the way down to the physical threshold $l=3$. Bray[17] has shown that an $F(u)$ constructed in this way offers a convincing interpolation between the known $D=2$ and $4-\epsilon$ threshold functions. For $|\alpha|\ll 1$ it is possible to linearize with respect to this parameter as well as $\eta$, giving

$$F(u)=1-u^{-3/2}(1+B\ln u), \tag{11}$$

with $0<B\leq 1.5$. Equation (11) is shown by the solid curves in Fig. 1 for $B=0.5$, 1.0, and 1.5. The unphysical case of $B=0$ is also included for comparison, although this would correspond to an absence of critical specific heat. Carrying out the integration in Eq. (10) gives

$$\eta=\frac{9}{75+104B}, \tag{12}$$

so that the upper and lower extreme values for $\eta$ are 0.12, and 0.039. The intermediate values are 0.071 ($B=0.05$) and 0.051 ($B=1.0$). Bray's spectral function, shown by the dashed curve in Fig. 1, is approximately halfway in between our $B=1.0$ and $B=1.5$ curves. Therefore his spectral function is consistent with $\eta\approx 0.044$.

In summary, we have pointed out the existence of a new sum rule, imposed by the critical-region temperature dependence of the specific heat. According to this sum rule, the Fisher-Langer tail in the critical correlation function must have a shape and strength such as to cancel out the Ornstein-Zernike hole exactly. We have shown how the sum rule can be translated into the language of the spectral function, thereby yielding a unique value for $\eta$. In other words, any other value of $\eta$ for the same spectral function would violate the sum rule and give the wrong critical specific-heat temperature dependence. Spectral functions of the right shape correspond to $\eta\approx 0.04$.

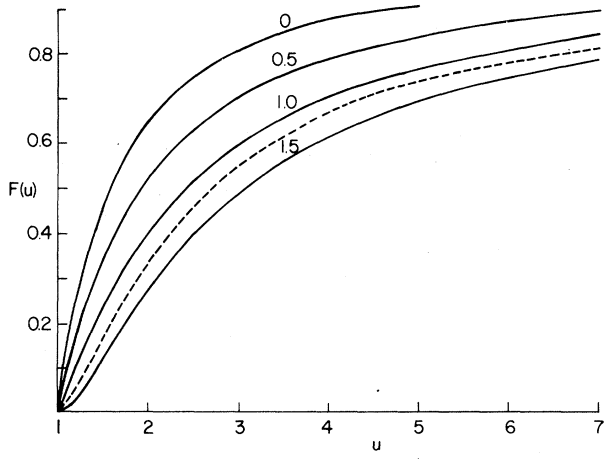


FIG. 1. Spectral function for $B=0$, 0.5, 1.0, and 1.5 vs imaginary wave number (in units of the threshold value). The dashed curve shows Bray's spectral function. The lowest curve ($B=1.5$) corresponds to $\eta=0.039$.

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# $n$-$n$ Scattering Length from the Photon Spectra of the Reactions $\pi^- d \rightarrow \gamma nn$ and $\pi^- p \rightarrow \gamma n$

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A high-statistics photon spectrum from the reaction $\pi^- d \rightarrow \gamma nn$, measured with a 0.72-meV–resolution pair spectrometer is compared with different theoretical spectra folded with the measured photon line from the reaction $\pi^- p \rightarrow \gamma n$. A value of $a_{nn} = -18.5 \pm 0.5$ fm (theoretical error included) is deduced assuming a fixed value of $r_{nn} = 2.8 \pm 0.1$ fm. The problem of extracting $r_{nn}$ using the present theories is indicated.

The current value of the singlet $^1S_0$ scattering length $a_{nn}$, based on an average of selected experiments,[1] is $-16.6 \pm 0.6$ fm. This is smaller in absolute value than $a_{pp}$(nuclear) $= -17.1 \pm 0.2$ fm deduced from low-energy $p$-$p$ scattering after correction for the direct electromagnetic effects. However, a charge-symmetry breaking due to isospin mixing in the exchange of vector mesons ($\rho$-$\omega$ mixing) indicates[2] an absolute value of $a_{nn}$ 0.9 fm larger than $|a_{pp}|$. This discrepancy could be explained by the model-dependent corrections of the direct electromagnetic effects[3] or an underestimate of the theoretical error in extracting $a_{nn}$. A remeasurement of $a_{nn}$ with higher experimental accuracy using a reliable theory seems therefore necessary to clarify the situation.

$\pi^- d \rightarrow \gamma nn$ is one of the most suitable reactions to study the $n$-$n$ final-state interaction because it involves no other hadron in the final state. The photon energy is uniquely related to the $n$-$n$ relative momentum $p$. To be sensitive mainly to the scattering length and to minimize the theoretical uncertainties, a measurement should be restricted to $p < 35$ MeV/$c$; i.e., to photons with energies within the last MeV of the spectrum, which ends at 131.458 MeV. Thus, the spectrometer resolution is important. The limitation on $p$ can be much more easily achieved by detecting the two neutrons. An experiment of this type by Salter *et al.*[4] yielded $a_{nn} = -16.7 \pm 1.3$ fm, not including the theoretical uncertainty of ~ 1 fm evaluated by Bander[5] for the extraction of $a_{nn}$ with his theory. However, precision experiments with neutrons in the few-MeV energy range are notoriously difficult, which makes the simpler photon spectroscopy attractive.

Two recent calculations[6,7] demonstrate the possibility of extracting $a_{nn}$ to an accuracy of 0.3 fm if $p < 35$ MeV/$c$. If this accuracy is to be reached in an experiment detecting only photons, a major difficulty arises. Since not only the shape of the photon spectrum but also the peak position is sensitive to $a_{nn}$, an energy calibration better than 3 keV at 130 MeV is needed to obtain $a_{nn}$ with a similar experimental error. This difficulty, however, can be overcome if the 129.406-MeV calibration line from the $\pi^- p \rightarrow \gamma n$ reaction and the $\pi^- p \rightarrow \gamma nn$ spectrum are measured alternately. The position of the line may be defined to within 1 keV with 100 000 events and the 720-keV full width at half maximum resolution of our spectrometer. Since this calibration line also provides the response function, with which the theoretical spectrum has to be folded, the only criti-

 