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Continuous-Spin Ising Model and $\lambda{:}\varphi^4{:}_d$ Field Theory

George A. Baker, Jr., and John M. Kincaid

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545

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Analysis of the high-temperature series expansions of a continuous-spin Ising model (i.e., a lattice $\lambda{:}\varphi^4{:}_d$ field theory) indicates that the strong-coupling limit of $\lambda{:}\varphi^4{:}_3$ is distinct from the critical-point theory of the continuous-spin Ising model for sufficiently Ising-like spin distributions. In four dimensions we find that the $\lambda{:}\varphi^4{:}$ field theory is trivial.

As suggested by Symanzik,[1] certain model field theories are mathematically similar to a class of statistical mechanical systems when the real time is replaced by imaginary time. The usefulness of these similarities was developed considerably by Wilson.[2] He pointed out that the removal of the ultraviolet cutoff in field theory was closely related to the approach to the critical point in the statistical mechanics of critical phenomena and then he adapted the machinery of the field-theoretic renormalization group to the analysis of the critical point.[3] The assumption that an ultraviolet-cutoff field theory possesses a unique, finite strong-coupling limit, independent of the manner in which the cutoff is removed and the strong-coupling limit is taken, is crucial to this approach. If this implicit assumption holds, then it is the thrust of the field-theoretic approach that the critical behavior of a whole class of systems is described by that strong-coupling limit.

We find that the unique-limit hypothesis fails in three and four dimensions. This failure was foreshadowed by the failure of hyperscaling in the $s=\frac{1}{2}$ Ising model.[4,5] Our calculations are based on a high-temperature series expansion of the continuous-spin model defined below. Analysis of the series using integral approximant techniques[6] shows that the renormalization-group theory can be thought of as the theory of the maximum value (over all nonnegative, bare coupling constants) of the renormalized coupling constant. We find that in one and two dimensions this theory describes the critical behavior of the continuous-spin Ising model and is the strong-coupling limit of the corresponding field theory. In three dimensions the theory still describes the strong-coupling limit of the field theory, but is distinct from the critical-point theory of the continuous-spin model for sufficiently Ising-like spin distributions. In four dimensions the $\lambda{:}\varphi^4{:}$ field theory is found to be trivial (in accord with the less extensive calculations of Wilson and Kogut[3]), but apparently distinct from the critical behavior of the continuous-spin Ising model. The occurrence of a fundamental length, however, would permit nontrivial physical scattering. The ϵ-expansion[3] and the coupling-constant-expansion methods[7] are not in the least inconsistent with our results although the class of systems to which they are applicable is restricted. We know of no rigorous results which are in any way at variance with our conclusions. In our calculations on this problem we have added several high-temperature series terms to those known for the $s=\frac{1}{2}$ Ising model and these terms are listed herein.

Specifically, the model we choose to study is a lattice cutoff $\lambda:\varphi^4:_d$ field theory defined by

$$Z=M^{-1}\int_{-\infty}^{\infty}\cdots\int\prod_{\vec{j}}d\varphi_{\vec{j}}\exp\Big[-\frac{1}{2}v\sum_{\vec{i}}\Big\{\frac{2d}{q}\sum_{\{\vec{\delta}\}}\frac{(\varphi_{\vec{i}}-\varphi_{\vec{i}+\vec{\delta}})^2}{a^2}+m^2\varphi_{\vec{i}}^2+2g_0(\varphi_{\vec{i}}^4-6C\varphi_{\vec{i}}^2+3C^2)+\delta m^2(\varphi_{\vec{i}}^2-C)\Big\}-\sum_{\vec{i}}H_{\vec{i}}\varphi_{\vec{i}}\Big], \quad (1)$$

where $v\propto a^d$ is the specific volume per site, a is the lattice spacing, d is the spatial dimension, q is the lattice coordination number, the sum over $\{\vec{\delta}\}$ is the sum over half the nearest-neighbor sites, m is the renormalized mass, δm^2 is the mass correction, $H_{\vec{i}}$ is the source field at site $\vec{i}$, g_0 is the bare coupling constant, and M is a normalization constant. The bracketed terms involving C are the normal ordered products $:\varphi^4:$ and $:\varphi^2:$, where C is the usual $[\varphi^-,\varphi^+]$ commutator (a sum over the lattice Green's functions) and tends to infinity as $a\to 0$ for $d\geq 2$. We may reexpress Eq. (1) in the form of the partition function of the continuous-spin Ising model by the change of variables $\sigma_{\vec{i}}=[2dv/qKa^2)]^{1/2}\varphi_{\vec{i}}$ so that Z becomes

$$Z=\tilde{M}^{-1}\int_{-\infty}^{\infty}\cdots\int\prod_{\vec{j}}d\sigma_{\vec{j}}\exp\Big[\sum_{\vec{i}}\Big\{K\sum_{\{\vec{\delta}\}}(\sigma_{\vec{i}}\sigma_{\vec{i}+\vec{\delta}})-\tilde{g}_0\sigma_{\vec{i}}^4-\tilde{A}\sigma_{\vec{i}}^2+\tilde{H}_{\vec{i}}\sigma_{\vec{i}}\Big\}\Big], \quad (2)$$

$$I_n(\tilde{H})=\int_{-\infty}^{\infty}dx\,x^n\exp[-\tilde{g}_0x^4-\tilde{A}x^2+\tilde{H}x]\{\int_{-\infty}^{\infty}dx\exp[-\tilde{g}_0x^4-\tilde{A}x^2+\tilde{H}x]\}^{-1}. \quad (3)$$

This condition determines $\tilde{A}$ as a function of $\tilde{g}_0$ alone and we shall only consider in this Letter functions that are independent of the σ scale. Note that for $\tilde{g}_0=0$ one obtains the Gaussian model while for $\tilde{g}_0=\infty$ one obtains the $s=\frac{1}{2}$ Ising model.

We will be concerned with the thermodynamic quantities

$$M=\langle\sigma_0\rangle,\quad \chi=\sum_{\vec{j}}\langle\sigma_0\sigma_{\vec{j}}\rangle,\quad \xi^2=[\sum_{\vec{j}}(\vec{j})^2\langle\sigma_0\sigma_{\vec{j}}\rangle](2d\chi)^{-1},$$
$$\left(\frac{\partial^2\chi}{\partial\tilde{H}^2}\right)_K=\sum_{\vec{j},\vec{k},\vec{l}}\langle\sigma_0\sigma_{\vec{j}}\sigma_{\vec{k}}\sigma_{\vec{l}}\rangle-\langle\sigma_0\sigma_{\vec{j}}\rangle\langle\sigma_{\vec{k}}\sigma_{\vec{l}}\rangle-\langle\sigma_0\sigma_{\vec{k}}\rangle\langle\sigma_{\vec{j}}\sigma_{\vec{l}}\rangle-\langle\sigma_0\sigma_{\vec{l}}\rangle\langle\sigma_{\vec{j}}\sigma_{\vec{k}}\rangle, \quad (4)$$

where M is the magnetization per spin, χ is the magnetic susceptibility, ξ is the second moment definition of the correlation length, and angular brackets denotes the thermal average defined by the partition function of Eq. (2). We concentrate our study on the quantity

$$g=-\left(\frac{v}{a^d}\right)\left(\frac{\partial^2\chi}{\partial\tilde{H}^2}\right)_K\frac{1}{\chi^2\xi^d}, \quad (5)$$

which is basically the dimensionless renormalized coupling constant. Using the relation[8] $m^2a^2\xi^2=1$, we have for small $\tilde{g}_0$ that $g=g_0m^{d-4}+O(g_0^2)$. For convenience in what follows, we will use the mass renormalization $m=1$. From the point of view of critical phenomena in the continuous-spin Ising model, g is a diagnostic for the hyperscaling relations since for $T\to T_0$, $T>T_0$, we have

$$g\sim(T-T_c)^{\gamma+d\nu-2\Delta}. \quad (6)$$

Here T is the temperature ($\sim 1/K$), T_c is the critical temperature, and the critical indices γ, ν, and Δ have their usual meaning: γ is the susceptibility index, ν the correlation length index, and Δ the "gap" index.[5] Rigorously it is known that the limit of g as $T\to T_c^+$ is bounded,[9] but g may tend to zero. Finite g corresponds to the vanishing of the exponent in Eq. (6), i.e., the hyperscaling relation $\gamma+d\nu-2\Delta=0$ holds, and $g\to 0$ corresponds to the failure of hyperscaling. From the field-theoretic point of view, $g\to 0$ corresponds to a theory that has no scattering and is, therefore, trivial.[10]

Our approach to the analysis of this model is to compute the high-temperature expansions through tenth order of M, χ, $(\partial^2\chi/\partial\tilde{H}^2)_K$, and ξ^2 in powers of K by the method of Wortis.[11] The coefficients of the series are polynomials in the moments of the spin distribution, I_n. Our series results considerably extend the earlier work of Camp and Van Dyke.[12] The production of the series required the 6390 singly rooted, multiple-line star graphs and the 1099 multiple-line star graphs with exactly two odd vertices. All graphs have no more than ten lines. These graphs were generated from a basic list of 185 unrooted single-line star graphs with no more than ten lines. The number of graphs, in order of increasing cyclotomic index (given in parentheses), with l lines is given by the following: $l=1$, (1); $l=3$, (0,1);

$l=4$, (0, 1); $l=5$, (0, 1, 1); $l=6$, (0, 1, 2, 1); $l=7$, (0, 1, 3, 3); $l=8$, (0, 1, 4, 9, 2); $l=9$, (0, 1, 6, 20, 14, 1); $l=10$, (0, 1, 7, 40, 50, 12, 1). We have counted the free multiplicity and second moments $(\sum_{\vec{j}} r_{0\vec{j}}^2)$ of these graphs on the linear chain, plane square, triangular, simple cubic, body-centered cubic, face-centered cubic, hyper-simple cubic, and hyper-body-centered cubic lattices.[13] The zero-field moments, $I_n(0)$, were calculated numerically making use of the recursion relation $R_n = I_{2n+2}(0)/I_{2n}(0)$, $R_{n+1} + \tilde{A}/(2\tilde{g}_0) = (2n+1)/(4\tilde{g}_0 R_n)$, and the asymptotic analysis of Wehner and Baeriswyl.[14] The symbolic computations on algebraic data were accomplished using the ALTRAN system as implemented on a CDC 7600 computer.[15]

The series were analyzed using the integral approximant method[6] that is a generalization of the well-known Padé methods. Here the coefficients of the polynomials Q_M, P_L, and R_N are determined by the accuracy-through-order principle applied to

$$Q_M(x)(df/dx) + P_L(x)f(x) + R_N(x) = O(x^{L+M+N+2}) \quad (7)$$

from the series coefficients of the function $f(x)$. The resultant approximant $[N/L;M]$ behaves like $\varphi_1(x)(x-x_0)^{-\gamma} + \varphi_2(x)$ near a singular point x_0, where φ_1 and φ_2 are regular, with some special exceptions. Since $\xi^2 = qK/2d + O(K^2)$, we have reverted that series to give $K(\xi^2)$ and substituted it in the others to yield $\chi(\xi^2)$ and $(\partial^2\chi/\partial\tilde{H}^2)_K(\xi^2)$. We apply the integral approximants to $(\partial^2\chi/\partial\tilde{H}^2)_K/\chi^2$ as a function of $x = 2(d+1)\xi^2/[1+2(d+1)\xi^2]$ which moves the critical point at $\xi^2=\infty$ to $x=1$, and generally moves all other singularities outside the unit circle. The direct use of this approximation method allows the simultaneous computation of the value of the function and its critical index. The critical index is $d/2$ for our case if the hyperscaling relation holds and is less than $d/2$ otherwise.

Our results for $d=1$ and 2 indicate that $g(G_0)$, where $G_0 = g_0 a^{4-d}/(10 + g_0 a^{4-d})$, rises in a smooth monotonic way. We do not note, nor had we expected, any significant deviation of critical index from its expected value. The $d=1$ curve is consistent with that of Marchesin[16] obtained directly for $\xi^2=\infty$. The $d=2$ curve and the validity of hyperscaling agrees with expectations based on the work of Kadanoff.[17] Our results for $d=3$ and 4 are illustrated in Fig. 1 where we have plotted g as a function of G_0 for the bcc and hbcc lattices with $\xi^2=10^6$. For $d=3$ the value of the critical index is relatively steady and consistent with the

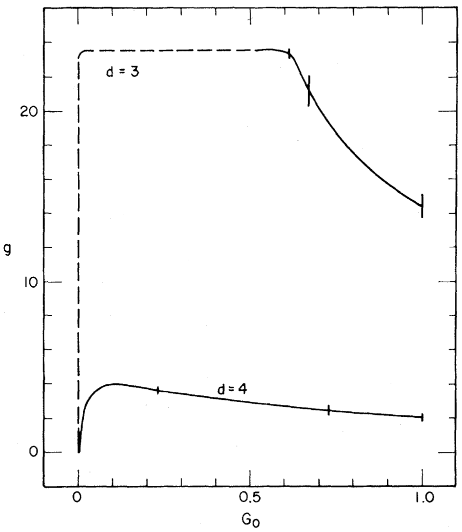

FIG. 1. The dimensionless renormalized coupled constant, g, on the bcc ($d=3$) and hbcc ($d=4$) lattices as a function of $G_0 = g_0 a^{4-d}$) for a large but fixed cutoff, $\xi^2 = 10^6$. The vertical bars indicate the apparent error in various regions. The curve for $d=3$ in the dashed region is basically beyond the range of good convergence of our approximants with only ten terms. It has been supplied from the convergent behavior of the g vs g_0 plots for smaller values of ξ^2.

corresponding Ising result[5,18] $\gamma + d\nu - 2\Delta = 0.028 \pm 0.003$, over the range $80 \lesssim g_0 a^{4-d} \leq \infty$. When one examines the behavior of g as a function of ξ^2 for fixed G_0, one finds a precipitous drop for very large values of ξ^2. This behavior corresponds to the small value of the index just quoted. The peak height (23.6) for $d=3$ in Fig. 1 agrees within error with the value $g^* = 1.420$ [units differ by a factor of $3/16\pi$ from ours],[7] and for fixed large G_0 the rapid descent begins from about this same value. When we replot the $d=1$, 2, and 3 curves as a function of g_0 alone we produce a nonperturbative calculation of $\lambda:\varphi^4:_d$ field theory. As a goes to zero the entire range $0 < g_0 < \infty$ collapses to the point $G_0=0$, and thus only the monotonically increasing portion of these curves is relevant for field theory and the peak appears to be the strong-coupling limit, as expected. The shape of the peak is consistent with the expectations of the Callan-Symanzik-equation approach.[7]

In four dimensions, over a range $100 \lesssim g_0 a^{4-d} \leq \infty$, we observe a relatively steady value of the critical index which corresponds to the Ising result[5] of $\gamma + d\nu - 2\Delta = 0.30 \pm 0.04$. Naturally $g \to 0$ as $\xi^2 \to \infty$ in this region. We have studied the behavior of the peak height (see Fig. 1) as a function of cutoff, and we find that its behavior is consistent with the idea that it shrinks like $1/\ln(\xi^2)$ as $\xi^2\to\infty$. One can easily deduce this behavior from the perturbation series $g=g_0-\gamma\ln(\xi^2)g_0^2+O(g_0^3)$. Consequently, we conclude that in three and four dimensions (at least for large $g_0 a^{4-d}$) hyperscaling fails and $\lambda:\varphi^4:_4$ field theory is trivial.

We have added for the $s=\frac{1}{2}$ Ising model the terms $5\,765\,546\,236\,416K^9/9!+271\,060\,330\,512\,384K^{10}/10!$ to the series for $2d\chi\xi^2$ (commonly referred to as μ_2) on the triangular lattice.[19] For the $(\partial^2\chi/\partial\tilde{H}^2)_K$ Ising series we have added the terms

$$-298\,834\,578\,777\,071\,616K^9/9!-39\,510\,128\,291\,537\,117\,184K^{10}/10!$$

on the fcc lattice,[20] the term $-601\,493\,660\,302\,278\,656K^{10}/10!$ on the hyper-simple cubic,[5] and the new series

$$-2-128K-9792K^2/2!-886\,784K^3/3!-92\,722\,944K^4/4!-11\,014\,965\,248K^5/5!$$
$$-1\,465\,369\,976\,832K^6/6!-215\,937\,597\,784\,064K^7/7!-34\,916\,329\,300\,783\,104K^8/8!$$
$$-6\,147\,843\,514\,432\,913\,408K^9/9!-1\,170\,908\,043\,876\,450\,435\,072K^{10}/10!$$

on the hbcc.

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