other nearby poles at $y = \pm i\gamma$ where $\gamma = \pi/2\mu$ provide the exact expression for the energy.[10] Other pole residues vanish exponentially as $\Lambda \to \infty$. Defining the physical mass[11]

$$m = m_0\left(\frac{e^{(1-\gamma)\Lambda}}{\pi(\gamma-1)}\tan\pi\gamma\right), \tag{18}$$

we find

$$E_n = m\cosh\gamma\alpha_1 + m\cosh\gamma\alpha_2, \quad n = r+1, r+2, \tag{19}$$

$$E_n = 2m\sin[\tfrac{1}{2}n\pi(2\gamma-1)]\cosh\gamma\alpha_s, \quad n \le r. \tag{20}$$

In the rest frame $\alpha_s = 0$, Eq. (20) gives the familiar sine-Gordon doublet spectrum of Ref. 3. The constant μ is related to the g of Ref. 1 by $2\mu = \pi(2g+\pi)/(g+\pi)$. By a similar calculation, the momentum is

$$P_n = m\sinh\gamma\alpha_1 + m\sinh\gamma\alpha_2, \quad n = r+1, r+2, \tag{21}$$

$$P_n = 2m\sin[\tfrac{1}{2}n\pi(2\gamma-1)]\sinh\gamma\alpha_s, \quad n \le r. \tag{22}$$

We have described an exact diagonalization of the massive Thirring model Hamiltonian. The method presents attractive possibilities for further study of the Thirring model as well as other field theories which are proven or conjectured to have an infinite number of conservation laws. The explicit expressions for eigenstates, Eqs. (2)–(4), provide a new approach to the study of Green's functions, reducing the question to a difficult but perhaps tractable problem of calculating inner products of Bethe wave functions.

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[10] Our cutoff procedure encounters difficulties for $\mu \le \frac{1}{3}\pi$ which will be discussed elsewhere. Here we always assume $\mu > \frac{1}{3}\pi$.

[11] Here, the limit $\Lambda \to \infty$ can be related to the lattice continuum limit discussed by Luther (Ref. 6) where the cutoff factor $e^{-\Lambda}$ is analogous to the elliptic modulus l^2 of the eight-vertex model and XYZ spin-chain formalism. The precise connection between l^2 and Λ emerges from a study of the critical limit of the eight-vertex model which will be presented elsewhere.

Ultraviolet Finiteness of All Quantum Loops in Gauge Supersymmetry

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The arbitrary n-point Green's functions of spontaneously broken gauge supersymmetry are shown to be ultraviolet finite to arbitrary loop order for $N \geq 2$ (where $4N$ is the number of Fermi coordinates) when the spontaneous breaking preserves global supersymmetry.

As is well-known, supersymmetry helps to reduce the ultraviolet infinities inherent in relativistic quantum field theories. Thus theories based on global supersymmetry have their ultraviolet infinities softened so that in many models only wave-function renormalizations are divergent.[1] In supergravity, the *S matrix* has been shown to be finite up to one- and two-loop order[2] (though there is some doubt as to whether higher loops are also finite[3]). The global supersymmetry of the S matrix also plays a fundamental role in canceling the S-matrix infinities of supergravity. In this note we will show that the quantum loops to *all orders* of gauge supersymmetry are finite when the theory possesses a spontaneous breaking which is globally supersymmetric.[4] Unlike supergravity where only S-matrix elements are finite, here the *off-shell Green's functions* are finite. Thus gauge supersymmetry represents the first example of a four-dimensional relativistic quantum field theory based purely on local gauge principles which is completely finite.

To maintain manifest global supersymmetry invariance, it is convenient to choose a linearized harmonic gauge-fixing term in the presence of a background metric $g_{AB}{}^{(0)} \equiv \langle 0 | g_{AB} | 0 \rangle$. Here $g_{AB}(z)$ is the single gauge field of gauge supersymmetry and $z^A = (x^\mu, \theta^{\alpha a})$, $\alpha = 1, \ldots, 4$, $a = 1, \ldots, N$ is the superspace coordinate. The anticommuting Fermi coordinate $\theta^{\alpha a}$ is labeled by a Majorana spinor index α, and an internal-symmetry index a. In the following we will compress the notation and write θ^α, $\alpha = 1, \ldots, 4N$. The vacuum transition amplitude then reads[4]

$$Z = \int dg_{AB} d\eta_A d\eta^{A*} e^{iI}. \tag{1}$$

Here $I = I_{\rm inv} + I_C + I_G$, where

$$I_{\rm inv}(g_{AB}) = \int dz(-g)^{1/2}[(-1)^B g^{BA} R_{AB} + (4N-2)\lambda], \tag{2a}$$

$$I_C = -\tfrac{1}{2}\int dz C^A C_A \text{ with } C^A \equiv g^{AB(0)} C_B,$$

$$C_A = g_{AB|C} g^{(0)CB} - \tfrac{1}{2}(-1)^{a+b}(g_{BC} g^{(0)CB})_{|A}, \tag{2b}$$

and

$$I_G = \int dz\, \eta^{A*}[\eta_{(A;B)|C} g^{(0)CB} - \tfrac{1}{2}(-1)^{a+b}(\eta_{(B;C)} g^{(0)CB})_{|A}], \tag{2c}$$

where $\eta_{(A;B)} \equiv \eta_{A;B} + (-1)^{a+b+ab}\eta_{B;A}$. In the above $\eta_A(z)$ and $\eta^{A*}(z)$ are the Faddeev-Popov ghost superfields; the subscript ";" means covariant derivative with respect to the full metric g_{AB}, while "|" is with respect to the vacuum metric $g_{AB}{}^{(0)}$. R_{AB} is the contracted curvature formed from g_{AB}.

It is clear that the above quantization maintains manifest global supersymmetry provided $g_{AB}{}^{(0)}$ is a global tensor. This implies $g_{AB}{}^{(0)}$ has the general form[4] $g_{\mu\nu}{}^{(0)} = \eta_{\mu\nu}$; $g_{\mu\alpha}{}^{(0)} = -i(\bar\theta\Gamma_\mu)_\alpha$; and $g_{\alpha\beta}{}^{(0)} = \eta_{\alpha\beta} + (\bar\theta\Gamma_\mu)_\alpha(\bar\theta\Gamma^\mu)_\beta$, where $\eta = -C^{-1}$ (C is the charge-conjugation matrix) and $\Gamma^\mu = \gamma^\mu\Gamma$ is a constant matrix in Dirac and internal-symmetry space. Writing $g_{AB}(z) = g_{AB}{}^{(0)} + h_{AB}(z)$ and expanding I in powers of h_{AB} allows one to construct the unperturbed propagators and vertices of the theory. Our procedure of proof is the following: We first determine in section I the structure of the unperturbed propagators required by global supersymmetry and their asymptotic behavior. The proof of finiteness of the simplest one-loop case of the two-point function is given in section II. (This illustrates the procedure, and allows the verification of some useful lemmas.) The analysis of finiteness is extended to the n-point one-loop case in section III. The overall degree of divergence D of the arbitrary n-point diagram with m loops is then shown in section IV to obey $D<0$. Since the subintegrations all have $D<0$ also (by section III), the finiteness of the arbitrary diagram follows as a consequence of Weinberg's theorem.[5]

(I) *Propagators.*—The equations for the tensor field propagator

$$\Delta_{ABCD}(z, z') = i\langle h_{AB}(z) h_{CD}(z')\rangle \tag{3}$$

and the vector ghost propagator $\Delta_A{}^B = i\langle \eta_A(z)\eta^{B*}(z')\rangle$ can be obtained to zeroth order from the quadratic parts of I. In the harmonic gauge these take the simple form

$$-\Delta_{ABCD|E}{}^E - 2\lambda\Delta_{ABCD} = [(-1)^{b+c+bc} g_{AC}{}^{(0)} g_{BD}{}^{(0)} + (-1)^{b+c+ca+ab} g_{BC}{}^{(0)} g_{AD}{}^{(0)} + (2N-1)^{-1} g_{AB}{}^{(0)} g_{CD}{}^{(0)}]\,\delta(z-z') \tag{4}$$

and

$$-\Delta_A{}^B{}_{|C}{}^C - (-1)^{ab+bc+ca}[\Delta_C{}^B{}_{|AD} - (-1)^{ad}\Delta_C{}^B{}_{|DA}] g^{DC(0)} = \delta^B{}_A\delta(z-z'). \tag{5}$$

Before attempting to solve these equations, it is convenient to impose the constraints of global supersymmetry invariance. Thus for the tensor propagator one has[6]

$$\Delta_{ABCD}(\overleftarrow{D}_\alpha + \overleftarrow{D}_{\alpha'}) + [\Delta_{AB(C\mu}\delta^\beta{}_{D)} + (-1)^{C+D}\Delta_{(A\mu CD}\delta^\beta{}_{B)}](i\eta\Gamma^\mu)_{\beta\alpha} = 0. \tag{6}$$

Here $\overleftarrow{D}_\alpha = \overleftarrow{\partial}_\alpha - i\overleftarrow{\partial}_\mu(\bar\theta\Gamma^\mu)_\alpha$ is the covariant derivative for the global transformation ξ^A ($\xi^\mu = i\bar\lambda\Gamma^\mu\theta$, $\xi^\alpha = \lambda^\alpha$), where λ^α is an infinitesimal constant anticommuting spinor. Equations (6) are a set of coupled equations for the components of Δ_{ABCD} which may be solved in a fashion similar to the scalar superfield analysis of Ref. (1). One finds in momentum space the result

$$\Delta_{ABCD}(k, \theta^\alpha, \theta^{\alpha\prime}) = \exp(\bar\omega\Gamma^\mu k_\mu\xi)\sum_{i=1,\ldots,4} F_{ABCD}{}^{(i)}(k, \omega^\alpha) P_{(i)}(\xi^\alpha), \tag{7}$$

where $\omega^\alpha \equiv \theta^\alpha - \theta^{\alpha'}$, $2\xi^\alpha \equiv \theta^\alpha + \theta^{\alpha'}$, and $P_{(i)}$ is a determined polynomial of ith degree and independent of p^μ and ω^α. Equation (7) contains the full constraint of global supersymmetry on Δ_{ABCD}. A similar form holds for $\Delta_A{}^B$ except the $P_{(i)}$ are at most quadratic in ξ^α.

One may now insert Eq. (7) into Eq. (6) and obtain equations for the $F_{ABCD}{}^{(i)}$ which may be solved in a power series in ω^α. The asymptotic form for large Euclidean p^μ is easily obtained for each power of ω^α and found to be[6,7]

$$F_{ABCD}{}^{(i)} \sim \sum_{n=0}^{4N} A_{(n)} k^{-(2+4N-n)}(\omega)^n, \tag{8}$$

where $(\omega)^n$ is short for $\omega^{\alpha_1}\cdots\omega^{\alpha_n}$. Thus the higher the power in ω^α, the more slowly the propagator vanishes in p^μ. A similar asymptotic form holds for the ghost propagator $\Delta_A{}^B$.

(II) *One-loop, two-point function.*—The interaction Lagrangian generated from Eqs. (2) by the translation $g_{AB} = g_{AB}{}^{(0)} + h_{AB}$ contains arbitrary powers of h_{AB} but each term contains precisely two superspace derivatives. The structure of the n-point momentum-space vertices is therefore the following: $V = V_2 + V_1\partial_\alpha + V_0\partial_\alpha\partial_\beta$, where V_a ($a=0, 1, 2$) are polynomials in momenta of order a. They may additionally depend on the θ^α variables of the lines into the vertex and ∂_α is short for the Fermi derivative of any of these θ^α. The key element of the vertices is that the *sum of the number of momentum factors plus ∂_α factors is less than or equal to 2.* This will be seen to effectively produce at most two momentum factors in each vertex.

Only the three- and four-point vertices enter into the one-loop, two-point integrals. The polarization operator has the form,

$$P_{ABCD}(p;\theta^\alpha,\theta^{\alpha'}) = \int d^4k\, V^{EFGH}_{AB(3)}(p,k;\theta)\Delta_{EFMN}(p+h;\theta,\theta')V^{MNRS}_{CD(3)}(p,k;\theta)\Delta_{RSGH}(h;\theta,\theta') + \int d^4k\, V^{EFGH}_{ABCD(4)}(p,h;\theta,\theta')\Delta_{EFGH}(k;\theta,\theta')\delta(\theta-\theta'). \tag{9}$$

We look first at the tadpole diagram involving $V_{(4)}$. The part of $V_{(4)}$ quadratic in k^μ, i.e., V_2, has no θ^α derivatives. Since $\delta(\theta-\theta')$ enforces $\theta^\alpha = \theta^{\alpha'}$ here, only the $n=0$ term of Eq. (8) survives in Δ. Thus the degree of divergence from the V_2 part of $V_{(4)}$ is $4+2-(2+4N)$ and so the integral converges provided that[8]

$$N \geq 2. \tag{10}$$

The $V_1\partial_\alpha$ part of $V_{(4)}$ contains at most one power of k^μ, but $\partial_\alpha\Delta$ allows $n=1$ in Eq. (8) which effectively supplies another power of k^μ, leading again to Eq. (10). All three parts of the vertex thus produces an $O(k^2)$ factor. This result is easily seen to be quite general and we will use it in all our following considerations.

Consider next the first integral of Eq. (9), which may be viewed as a function of ω^α and ξ^α. The most divergent part comes from the $(\omega)^{4N}$ coefficient. First, ignoring the exponential factor of Eq. (7) in Δ, the degree of divergence is $4+2(2)-(2+4N-n)-(2+4N-m)$, where $n+m \leq 4N$. Hence again Eq. (10) is a sufficient condition for convergence. The exponential of Eq. (7), when expanded, supplies one power of k^μ for each power of ω^α. Since the latter, as a consequence of Eq. (8), produces an effective $1/k$ factor, the exponential of Eq. (7) does not affect the convergence of the integral. Similarly, the polynomial factors $P_{(i)}(\xi)$ of Eq. (7) do not affect convergence as they are independent of ω^α.

III. *One-loop, n-point function.*—We generalize now to the case of a single loop containing n vertex points labeled $z_1, \ldots, z_n$ with $z_i{}^A = (x_i{}^\mu, \theta_i{}^\alpha)$. At each vertex an arbitrary number of external lines may be attached (since arbitrary-point vertices exist in the theory). The previous discussion has shown the following: (i) Each vertex effectively supplies a factor of $O(k^2)$. (ii) The exponential and polynomial $P_{(i)}(\xi)$ do not change the convergence properties. (iii) Ghost propagators behave as tensor propagators as far as convergence questions are concerned. Thus the convergence of the n-point diagram is governed by the integral

$$\int d^4k\, V_1F_1(p_1+k,\omega_{12})V_2F_2(p_2+k,\omega_{23}) \cdots V_nF_n(p_n+k,\omega_{n1}), \tag{11}$$

where p_i are functions of the external momenta and $\omega_{ij}{}^\alpha = \theta_i{}^\alpha - \theta_j{}^\alpha$. [In Eq. (11) we have compressed all superspace tensor indices into a single subscript.] Note that

$$\omega_{12} + \omega_{23} + \ldots + \omega_{n1} = 0, \tag{12}$$

i.e., the ω_{ij} are not all independent since global supersymmetry requires that the F functions in Eq. (7) depend only on the *differences* of θ^α coordinates. We now expand each F_i in Eq. (11) in powers of ω_{ij}. From Eq. (8), the degree of divergence of the coefficient of $(\omega_{12})^{n_{12}}(\omega_{23})^{n_{23}}\ldots$ is

$$D=4+n(2)-n(2+4N)+\sum n_{ij}. \quad (13)$$

Equation (12) implies that there are only $n-1$ independent ω_{ij} coordinates and so $\sum n_{ij}\leq(n-1)\times(4N)$ as each Fermi coordinate cannot appear with a power greater than $4N$. Hence $D\leq 4-4N$ and Eq. (10) again guarantees convergence.

IV. *m-loop, n-point function.*—We calculate next the overall degree of divergence of the arbitrary n-loop, m-point function. We may build up the arbitrary diagram by inserting internal lines into the one-loop, n-point function so that there are a total of p additional lines in the diagram and $s=m-1$ more loops formed. This will in general give rise to q new vertices (at points $z_1', \ldots, z_q'$, say) and hence q additional independent ω_{ij} Fermi variables (e.g., $\omega_r'\equiv\theta_1{}^{\alpha}-\theta_r{}^{\alpha'}$, $r=1,\ldots,q$). Momentum conservation at each of the new vertices implies $s=p-q$. Since the theory allows v-point elementary vertices, where $v=3, 4, \ldots,$ q can range from 0 to $2s$ (and hence $p=s, s+1, \ldots, 3s$). Thus the case $q=0$ corresponds to all the new lines ending on previously existing vertices (promoting each such v-point vertex to a higher vertex); $q=1$ corresponds to a new three- or higher-point vertex at the point z_1'; $q=2s$ corresponds to $2k$ new three-point vertices being inserted. We can now calculate the overall divergence of the diagram arising from the coefficient of $(\omega_{12})^{n_{12}}(\omega_{23})^{n_{23}}\cdots(\omega_{n1})^{n_{n1}}(\omega_1')^{n_1'}(\omega_2')^{n_2'}\cdots(\omega_r')^{n_r'}$. Since each vertex is $O(k^2)$ one has, as in Eq. (13),

$$D=(s+1)n_b+(n+q)(2)-(n+p)(2+n_f)+\sum n_{ij}+\sum n_r', \quad (14)$$

where $n_b=4$ is the number of Bose dimensions and $n_f=4N$ is the number of Fermi dimensions. Since all the internal Fermi coordinates are integrated over, only the terms with $n_r'=4N$ survive, while $\sum n_{ij}\leq(n-1)4N$ as before. One has then

$$D\leq m(n_b-n_f)-2(m-1) \quad (15)$$

since $p-q=s=m-1$. Thus $D<0$ for $N\geq 2$, and since the subintegrations have been seen to be convergent, the general m-loop graph is finite for $N\geq 2$, by Weinberg's theorem.[5]

V. *Additional comments.*—The gauge invariance of gauge supersymmetry is sufficiently powerful to uniquely determine all the dynamics of the theory. (Only the numerical value of the constant λ is undetermined.) It is remarkable that the two conditions of local gauge invariance and supersymmetry in gauge supersymmetry combine to produce a completely finite theory with no further *ad hoc* assumptions. Further, since the theory possesses no linear divergences for $N\geq 2$, one expects that it will be anomaly free.

An important remaining question in gauge supersymmetry is whether the theory possesses ghosts. It is now possible, for the first time, to examine this problem since loop corrections to the kinetic energy matrix can be calculated *and are finite.* One hopeful point in this connection is that usually one type of ghost pathology manifests itself by producing negative energy states when quantized with conventional positive-norm states. However, as is well-known, a theory possessing global supersymmetry (as is the case in the discussions here) must have only positive energy states.

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[6]A detailed discussion of the properties of the globally symmetric propagators described in this section will be given in a separate paper. The corresponding condition on the ghost propagator reads $\Delta_A{}^B(\overleftarrow{D}_\alpha+\overleftarrow{D}_{\alpha'})+[\Delta_A{}^\beta\delta_\mu{}^B-(-1)^b\Delta_\mu{}^B\delta_A{}^\beta](i\eta\Gamma^\mu)_{\beta\alpha}=0$.

[7]Actually, some components of F_{ABCD} fall off even faster than the bounds of Eq. (8). We will see, however, that Eq. (8) is sufficient to prove the finiteness of all Feynman graphs.

[8]It is possible that finiteness also exists for $N=1$ due to a cancellation of all the logarithmic infinities. We have not examined this possibility.