fields play an essential role, electrostatic models are insufficient. Since all three dimensions of the problem require roughly equivalent scale lengths, it is not possible to modify a 2D or $2\frac{1}{2}$D code by including a few modes in the third direction as is sufficient, for example, in the simulation of some tokomak properties.[3,4]

The principles of the simulation algorithms differ only slightly from those described by Buneman.[5] The use of Fourier transforms, quadratic spline interpolation [over a $(32)^3$ mesh], and careful charge shaping is retained, but the SPLASH code uses a different scheme for data management than that of C. Barnes described in Ref. 5. It is essential to keep most of the particle and field information on disk between time steps and care was taken to overlap input/output processes with computations. These techniques allow a simulation of this magnitude to be performed efficiently with the resources available through the NMFECC (National Magnetic Fusion Energy Computing Center) system. A more complete description of SPLASH, as well as the code itself, is available through the NMFECC on-line codeshare facility LIBRIS, maintained at Lawrence Livermore Laboratory, University of California, Livermore, California 94550.

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Stability of Field-Reversed, Force-Free, Plasma Equilibria with Mass Flow

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The existence of a special set of cusp-shaped, stable, force-free plasma equilibria with finite pressure and flow velocity is demonstrated. They are confined by surface currents.

In a series of papers Woltjer[1] has formulated the stability of hydromagnetic equilibria in terms of a variational principle in which the energy is minimized while keeping a number of integrals of motion of the system constant. Thus, if a system is able to dissipate energy, without changing these integrals of motion, it will pass to a state of minimum energy compatible with these constraints.[1] More recently, Taylor[2] has employed this variational technique to show that the minimum-energy state for a straight pinch bounded by a rigid, infinitely conducting, cylindrical container is a force-free helical configuration in which the axial field is reversed on the outside. For plasma confinement, however, one must insist on a vacuum region that separates the fluid boundary from the wall. Therefore, in addition to the internal fluid perturbations, the possibility of surface deformations has to be entertained. These have been included in an elegant treatment by Rosenbluth and Bussac,[3] that evaluates the stability of a zero-pressure, force-free spheromak.[4] Since high-beta toroidal plasma confinement systems are highly desirable,[5] we seek stable configurations in which the plasma pressure is finite. There is no guarantee, however, that fluid kinetic energy of motion in such systems will be negligible.[6] Indeed experiments exist[7,8] and others are planned[9] in which this kinetic energy is comparable to the magnetic energy. In this Letter I demonstrate the existence of stable equilibria with finite flow velocity that have force-free fields in the interior but are confined by surface currents.

The hydromagnetic equations, in principle, possess an infinite number of integrals of motion, and to recover all possible states of motion one needs to consider all of them. However, one can obtain interesting states of minimum energy even by considering a reduced set.

I depart from Woltjer's treatment by considering the plasma in the two-fluid instead of the hydromagnetic approximation. I choose the following system integral constants: the flux invariant $K=\frac{1}{2}\int d^3x\vec{A}\cdot\vec{B}$ ($\vec{B}=\nabla\times\vec{A}$, $\vec{B}$ is the magnetic field), the three components of the system angular momentum $\vec{L} = \int d^3x(\vec{r}\times nm\vec{v})$ and the total mass $Nm = \int nm d^3x$. The last constant is obtained as follows. The ion fluid momentum equation is

$$\partial\vec{v}/\partial t + \nabla v^2/2 - \vec{v}\times\nabla\times\vec{v} - (q/m)(\vec{E}+\vec{v}\times\vec{B}) + \nabla p/nm = 0, \tag{1}$$

where p, $\vec{v}$, and n are the pressure, fluid velocity, and density, respectively. I neglect the electron inertia and describe the electron fluid by $\vec{E}+\vec{u}\times\vec{B}=0$ and Faraday's law furnishes $\partial\vec{B}/\partial t = \nabla\times\vec{u}\times\vec{B}$, where $\vec{u}$ is the electron fluid velocity. Charge neutrality is assumed to hold. Multiplying (1) by $\vec{B}\cdot$ and integrating over the region occupied by the plasma with the following boundary conditions on the plasma surface

$$\vec{B}\cdot\hat{s} = \vec{v}\cdot\hat{s} = \vec{j}\cdot\hat{s} = 0, \tag{2}$$

where $\vec{j}$ is the plasma current and $\hat{s}$ is the unit vector normal to the plasma surface, I obtain

$$\partial G/\partial t \equiv \partial/\partial t \int d^3x\, \vec{v}\cdot[\vec{B} + (m/2q)\nabla\times\vec{v}] = 0. \tag{3}$$

In arriving at (3) I have assumed (a) that the pressure is a function of density or less restrictively, $\nabla p\times\nabla n = 0$ and (b) $\hat{s}\cdot\nabla\times\vec{v} = 0$ on the boundary at some initial instant. Then the relation

$$\{[\vec{B} + (m/q)\nabla\times\vec{v}]/n\} = \{[\vec{B}_0 + (m/q)\nabla\times v_0]/n\}\cdot\nabla_0\vec{r}, \tag{4}$$

guarantees that boundary condition (b) holds for all time; the subscript 0 indicates values at $t=0$. For a compressible fluid in which the pressure variations are adiabatic with index γ the system energy $W = \int d^3x[\frac{1}{2}nmv^2 + \frac{1}{2}B^2 + p/(\gamma-1)]$ is also constant. Keeping G, K, L, and the total mass $Nm = \int nm\, d^3x$ constant, one can minimize W; thus

$$\delta C = \delta W - \lambda\delta K - \mu\delta G - \vec{\nu}\cdot\delta\vec{L} - \eta\delta N = 0, \tag{5}$$

where the constants λ, μ, $\vec{\nu}$, and η are Lagrange multipliers. I arrive at the following system of equations describing the extremum-energy state analogous to Woltjer's[1] solutions,

$$\nabla\times\vec{B} + \lambda\vec{B} + \mu\nabla\times\vec{v}, \tag{6}$$

$$\mu\nabla\times\vec{v} = nq(\vec{v} - \vec{r}\times\vec{\nu}) - (\mu q/m)\vec{B}, \tag{7}$$

$$\eta = \tfrac{1}{2}mv^2 + \vec{\nu}\cdot\vec{r}\times\vec{v} + \gamma T/(\gamma-1), \tag{8}$$

where $p = nT$, and T is the kinetic temperature.

For incompressible flow, $\nabla\cdot\vec{v} = 0$, and furthermore, let $n = \text{const}$; Eq. (8) is equivalent to the Bernoulli condition

$$\nabla p + \tfrac{1}{2}nm\nabla v^2 = 0. \tag{9}$$

From Eqs. (6) and (7) by eliminating $\vec{B}$ in favor of $\vec{v}$ and vice versa we obtain,

$$\nabla\times\nabla\times\vec{v} - (q/m\mu)(nm - \mu^2 + \lambda\mu m/q)\nabla\times\vec{v} + (nq\lambda/\mu)\vec{v} = -(nq/\mu)\nabla\times\vec{r}\times\vec{\nu} - (nq\lambda/\mu)\vec{r}\times\vec{\nu} \tag{10}$$

and

$$\nabla\times\nabla\times\vec{B} - (q/m\mu)(nm - \mu^2 + \lambda\mu m/q)\nabla\times\vec{B} + (nq\lambda/\mu)\vec{B} = -nq\nabla\times\vec{r}\times\vec{\nu}. \tag{11}$$

The general solutions of Eqs. (10) and (11) are given by

$$\vec{v} = a\vec{h}_1 + b\vec{h}_2 - 2(\mu/nq)(1 - \mu q/m\lambda)\vec{\nu} + \vec{r}\times\vec{\nu} \tag{12}$$

$$\vec{B} = c\vec{h}_1 + d\vec{h}_2 + 2\mu\lambda^{-1}\nu, \tag{13}$$

where $\vec{h}_1$ and $\vec{h}_2$ are the general solutions of

$$\nabla^2\vec{h}_{1,2} + \kappa_{1,2}{}^2\vec{h}_{1,2} = 0, \tag{14a}$$

$$\kappa_{1,2} = \tfrac{1}{2}(q\mu/m)\{(\delta+\epsilon) \pm [(\delta-\epsilon)^2 - 4\epsilon]^{1/2}\}, \tag{14b}$$

$\epsilon \equiv \lambda m/\mu q$, $\delta + 1 \equiv (nm/\mu^2)$, and a, b, c, and d are constants. For $\nu = 0$, the solutions have no "rigid" rotation and the boundary conditions (2) lead to solutions in which the velocity is parallel to the field,

$$\vec{v} = \alpha\vec{B}, \tag{14c}$$

and $\alpha = (\kappa - \lambda)/\mu\kappa$. A nonzero ν imposes rather severe conditions on obtaining an equilibrium and we ignore this case in what follows.

In addition to (2) we have to satisfy the following condition on boundary surface between the plasma and the surrounding medium,

$$(p + \tfrac{1}{2}B^2)_{\text{I}} = (p + \tfrac{1}{2}B^2)_{\text{II}}, \tag{15a}$$

and $\vec{B}$ is tangential to the surface of discontinuity. If the external medium is vacuum $p_{\text{II}} = 0$ and if the boundary is a conducting wall then (15a) is not needed. From (9) we may write $p + \frac{1}{2}nmv^2 = P$ (const), and since $(nm)^{1/2}\vec{v} = \beta\vec{B}$ with β constant we may rewrite (15a) as

$$[P + \tfrac{1}{2}(1-\beta^2)B^2]_{\text{I}} = [p + \tfrac{1}{2}B^2]_{\text{II}}. \tag{15b}$$

The general solution of (14a) is given by $\vec{B} = \kappa\vec{r}\times\nabla\Phi + \nabla\times(\vec{r}\times\nabla\Phi)$ with $\nabla^2\Phi + \kappa^2\Phi = 0$. We derive some interesting properties of axisymmetric equilibria ($\partial/\partial\varphi = 0$). Let $\vec{B} = \vec{B}_t + \vec{B}_p$, where $\vec{B}_t = \hat{\varphi}\kappa\,|\vec{r}\times\nabla\Phi|$ is the toroidal component and $\vec{B}_p = \nabla\times(\vec{r}\times\nabla\Phi)$ is the poloidal component of the field. Thus,

$$\nabla\times\vec{B}_p = \kappa\vec{B}_t \text{ and } \nabla\times\vec{B}_t = \kappa\vec{B}_p. \tag{16}$$

The magnetic surfaces are defined by the poloidal flux $\psi = \kappa^{-1}\rho\hat{\varphi}\cdot\vec{r}\times\nabla\Phi = \kappa^{-1}\rho B_t$ and $\vec{B}\cdot\nabla\psi = 0$; here (r,θ,φ) and (ρ,φ,z) represent the coordinates of a spherical and cylindrical system, respectively. I construct a third coordinate system such that $\hat{x}_\parallel = \vec{B}_p/B_p$, $\hat{x}_\perp = \hat{x}_\parallel\times\hat{\varphi}$, $d\psi = d\vec{x}_\perp\cdot\nabla\psi$, $|\nabla\psi| = \rho B_p$ and the toroidal flux $\chi(\psi) = \kappa^{-1}\oint dx_\parallel B_p$. From (16) we obtain,

$$\int_V d^3x(B_t^2 - B_p^2) = \kappa^{-1}\int_S d^2x\, B_p B_t, \tag{17}$$

where S is the magnetic surface enclosing volume V. If region II outside the plasma is to be vacuum, B_t vanishes on the plasma boundary. In this case, I conclude from (17) that there is equipartition of energy between the poloidal and toroidal components of both field and fluid kinetic energy by virtue of (14c). Further manipulation of Eqs. (16) leads to

$$\tfrac{1}{2}\nabla(B_t^2 + B_p^2) = \tfrac{1}{2}\hat{x}_\parallel\partial B_p^2/\partial x_\parallel - \hat{x}_\perp B_p^2/\rho_p - \hat{\rho}B_t^2/\rho, \tag{18}$$

where $\hat{x}_\parallel\cdot\nabla\hat{x}_\parallel = -\hat{x}_\perp/\rho_p$, and $\rho = r\sin\theta$. From (18) one obtains

$$\tfrac{1}{2}\partial B_t^2/\partial x_\parallel = -B_t^2\hat{x}_\parallel\cdot\hat{\rho}/\rho, \tag{19a}$$

and

$$\tfrac{1}{2}\rho B_p\frac{d}{d\psi}(B_t^2 + B_p^2) = -\left(\frac{B_p^2}{\rho_p} + \frac{B_t^2}{\rho}\cos\delta\right), \tag{19b}$$

or

$$\tfrac{1}{2}\int_S d^2x(B_t^2 + B_p^2) = \int_V d^3x\left(\frac{B_p^2}{\rho_p} + \frac{B_t^2}{\rho}\cos\delta\right), \tag{19c}$$

where $\hat{\rho}\cdot\hat{x}_\perp = \cos\delta$ and S is a magnetic surface of volume V.

In vacuum $p_{\text{II}} = 0$ and a surface current $I = \hat{\varphi}B_{\text{I}}\{[(1-\beta^2) + 2P/B_{\text{I}}^2]^{1/2} - 1\}$ flows to confine the plasma. The vacuum magnetic field is computed from $\rho\vec{B} = \nabla\psi\times\hat{\varphi}$, and $\partial^2\psi/\partial\rho^2 - \rho^{-1}\partial\psi/\partial\rho + \partial^2\psi/\partial z^2 = 0$, with boundary conditions determined by (2) and (15b) on the plasma surface and by the windings generating the external field. However, since $\psi = \text{const}$ on the plasma surface and (15b) (which involves $\nabla\psi$) must also be observed, the elliptic equation for ψ is overdetermined. We are thus led to consider a free boundary problem in which the shape of the plasma surface is determined by the external field and the parameters κ, etc. We now show how an explicit solution is possible for the case $\beta = 1$, i.e., $\vec{v} = \vec{B}/(nm)^{1/2}$. From (15b) it is apparent that with $\beta = 1$, B_{II} is constant on the plasma surface. The problem thus reduces to an equivalent problem in fluid mechanics ($\vec{B}\to\vec{v}$), viz., Helmholtz flow in which fluid velocity is constant on the free streamlines (from Bernoulli's principle). In two dimensions, i.e., $\rho\to\infty$, the theory of conformal mapping[10] can be used to determine the shape of the free streamlines or equivalently the plasma boundary. The shape of the plasma boundary is determined by the field at infinity. If the field at large distances is uniform then the plasma boundary must be toroidal; a spheromak topology is disallowed. In three dimensions an exact solution is possible only for a plasma torus of large aspect ratio. Omitting here the details of the derivation, one finds that the plasma ring is circular in cross section. The field at infinity B_0 is related to the field on the plasma surface B_s through[11,12]

$$B_0 = (I/4\pi R)[\ln(16\pi R B_s/I) - \tfrac{1}{2}], \tag{20}$$

where I is the azimuthal surface current R and a the ring major and minor radii. To next order in a/R the cross section is D shaped (with the inner portion having a larger radius of curvature) given by,

$$\{1 - \tfrac{1}{4}[12\ln(4/\sigma) - 15]\sigma^2\cos 2\theta\}\sigma = \text{const}, \tag{21}$$

where $\sigma = \exp(-\Xi)$, and Ξ, θ, and φ are toroidal coordinates defined by $\Xi + i\theta = \ln[(\rho + iz + R)/(\rho + iz - R)]$. Equation (20) is valid to this order. The solutions for the interior are obtained, in principle, by solving $\nabla^2\Phi + \kappa^2\Phi = 0$, $\kappa \neq 0$, with $\psi = \kappa\rho\hat{\varphi}\cdot\vec{r}\times\nabla\varphi = 0$ on the boundary. $\vec{B}\cdot\nabla\psi = 0$ on the boundary thus ensuring that $\vec{B}\cdot\hat{s}$ also vanishes. Since the boundary is not likely in general to coincide with a coordinate system for which the Helmholtz equation is separable, in practice one would seek numerical solutions.

A quadrupole external field determines a cusp-shaped boundary. The internal fields form two vortex rings with $\vec{B}$ fields in the opposite sense (see Fig. 1). An m-pole field will give the plasma a free surface of m inverted cycloids, viz., the picket-fence geometry.[13]

The total energy W of the constant density, incompressible, equilibria with $\vec{v} = \beta\vec{B}/(nm)^{1/2}$ obtained through Eqs. (6) and (7) does not necessarily have a minimum in energy but only an extremum. To establish a true minimum I express W in terms of K and G, thus $W = K(nq^2/m)^{1/2}(\beta + 1/\beta)[-1 \pm (1 + |mG/qK|)^{1/2}]$ (signs are chosen to make $W > 0$). It is clear that W will be minimum for β^2

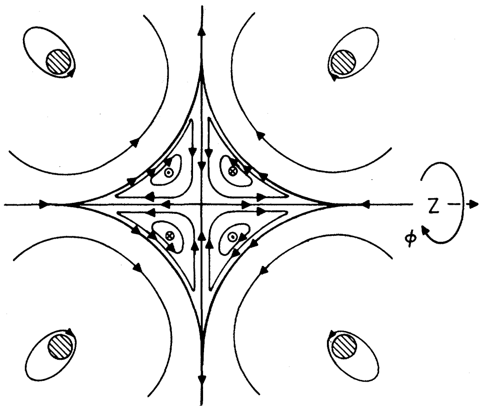

FIG. 1. A section in the poloidal (ρ, z) plane of a spindle cusp. External field is produced by coils (hatched) and fields within plasma are generated by two topological vortex rings with oppositely oriented toroidal fields.

$\rightarrow 1$, and stability against internal motion is assured. To establish stability against a surface displacement we follow Rosenbluth and Bussac[3] and perturb the surface by a displacement $\xi(x_\parallel, \varphi)\hat{x}_\perp$, allowing the interior to relax to a minimum-energy state for the new surface consistent with K and G being conserved.

Stability is determined by whether the pressure imbalance across the surface reinforces or opposes the displacement, i.e., if

$$\delta E = \int_{S_b} d^2x\, \xi \left\{ \vec{B}_{\rm II} \cdot \vec{B}_{\rm II}^{(1)} - \vec{B}_{\rm I} \cdot \vec{B}_{\rm I}^{(1)}(1-\beta^2) + \xi \frac{\partial}{\partial x_\perp}[\vec{B}_{\rm II}^2 - B_{\rm I}^2(1-\beta^2)] \right\} > 0, \quad (22)$$

we have a stable system; $\vec{B}_{\rm I}^{(1)}$ and $\vec{B}_{\rm II}^{(1)}$ are the perturbed fields in plasma and vacuum, respectively. For equilibria with $\beta = 1$ the influence of the internal magnetic fields is annihilated by the velocity perturbations and this expression simplifies to

$$\delta E = \int_{S_b} d^2x\, \xi\, (\vec{B}_{\rm II} \cdot \vec{B}_{\rm II}^{(1)} - \xi B_{\rm II}^2/\rho_b), \quad (23)$$

where ρ_b is the radius of curvature of the plasma boundary considered positive for a concave surface viewed from within the plasma. Now $\vec{B}_{\rm II}^{(1)}$ is the gradient of a harmonic function and $\hat{x}_\perp \cdot \vec{B}_{\rm II}^{(1)} = \hat{x}_\perp \cdot (\vec{B}_{\rm II} \cdot \nabla)\vec{\xi}$ at the boundary from the frozen-flux condition. Furthermore, $\vec{B}_{\rm II}$ is tangential to the surface and is constant. From these facts it can be established that the first term of δE is always positive. Therefore, for equilibria where ρ_b is everywhere negative, $\delta E > 0$. Thus, constant density force-free equilibria with cusp-shaped surface and with $\vec{v} = \vec{B}/(nm)^{1/2}$ are stable to both internal and surface modes.[14]

When, however, ρ_b is positive, as is the case for a toroidal plasma, then the question of stability has to be treated case by case. A thin ring, $R/a \gg 1$, degenerates into a Z pinch with a surface current with no contribution from the internal fields and is, therefore, unstable. The situation improves for fatter D-shaped rings but, numerical analysis is required to compute both the equilibria and stability.

Finally, I conjecture that a high-beta plasma probably evolves into a state such that there is a central force-free region surrounded by a skin region in which currents flow across field lines to sustain the pressure drop. In the equilibria discussed above, the skin layer has been idealized to be infinitesimally thin, but in practice it will, of course, have a finite thickness depending upon transport coefficients and the manner in which it has been created. The large kinetic energy needed for these configurations may require the injection of high-power intense ion beams,[9] or neutral beams, or by vortex-ring compression.[8]

The study of compressible equilibria will be taken up in a future publication. I am indebted to Professor M. N. Rosenbluth for many discussions and a preprint of his work on spheromak stability (see Ref. 3) presented at the Princeton Conference on Plasma Rings in June, 1978, and to Professor H. Weitzner for some useful comments. This work was supported in part by the U. S. Department of Energy.

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