magnitude as the one in the absence of the cooperative effects and therefore should be more directly accessible to experimental observation.

[15]If one considers a system with large numbers of atoms which are emitting independently, then one finds (Ref. 11)

$$\gamma^{(2)}(\tau) = 1 + \tfrac{1}{4}e^{-2\gamma\tau}(1 + \tfrac{1}{2}e^{-\gamma\tau} + \tfrac{1}{8}e^{-3\gamma\tau}\cos(4\Omega\tau) + \tfrac{1}{2}e^{-(5/2)\gamma\tau}\cos(2\Omega\tau).$$

Again, there is a large difference between the structure of $\gamma_{(\tau)}^{(2)}$ obtained under cooperative conditions and the above expression as far as the weights and decay rates of the various contributions are concerned.

Variational Method in Turbulence Theory

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The weighted mean square of the Navier-Stokes equation is mimimized with a complete set of realizability inequalities as constraints. Expansion of moments in complete orthogonal functions leads to successive approximations without ever involving moments of order higher than 4. Alternatively, the expansions may be in Wiener-Hermite kernels, thereby automatically satisfying the realizability constraints. The approach extends to other classical and quantized systems with polynomial nonlinearity.

The nonlinearity of the Navier-Stokes (NS) equation couples velocity-field moment equations of all orders. However, the values of moments up to fourth order are sufficient to determine whether an ensemble of velocity fields satisfy the NS equation in mean square. We suggest that this can be the basis of a closed formulation, useful for computation, in which moments of order higher than 4 never appear and the missing information contained in the initial values of the higher moments shows up as freedom in the evolution of the moments retained.

The NS equation may be written as

$$L_i(t) = dy_i/dt + \textstyle\sum_j \nu_{ij} y_j + \sum_{jk} A_{ijk} y_j y_k = 0, \qquad (1)$$

where the $y_i(t)$ are the real amplitudes of linearly independent modes of the velocity field, ν_{ij} is a damping matrix with positive eigenvalues, and $A_{ijk} + A_{jki} + A_{kij} = 0$ (conservation of $\sum_i y_i^2$ by the nonlinearity). Consider the limit $\kappa \to 0$ of the constraint on the ensemble[1]

$$\textstyle\sum_i \int_0^T \langle [L_i(t)]^2 \rangle \rho_i(t) dt = \kappa^2, \qquad (2)$$

where $\langle\,\rangle$ denotes ensemble average, and $\rho_i(t) > 0$ for $0 \leq t \leq T$. Equation (2) expresses approximation to the NS equation in mean square, with $L_i(t) = O(\kappa)$ in typical realizations. It implies, via Schwarz inequalities expressing positivity of the probability density, that the hierarchy of NS many-time moment equations is satisfied with errors $O(\kappa)$ as $\kappa \to 0$, if the moments of all orders exist in the limit. Equation (2) involves only second-, third-, and fourth-order moments. It can be used variationally to determine values of these moments that solve the NS equation provided the values are constrained to be realizable (i.e., represent an ensemble with nowhere negative probability density).

For a single real stochastic variable b the necessary and sufficient conditions for realizability are[2]

$$I_{2n}(\lambda) = \langle (\sum_{r=0}^{n} \lambda_r b^r)^2 \rangle$$

$$= \sum_{r,s=0}^{n} \lambda_r \lambda_s \langle b^{r+s} \rangle \geq 0 \quad (n = 1, 2, \ldots, \infty) \qquad (3)$$

for all real λ_s such that $\sum_{s=0}^{n} \lambda_s^2 > 0$. Equation (3) states that the symmetric matrix $Q_{rs} = \langle b^{r+s} \rangle$ has no negative eigenvalues. The moments $\langle b^n \rangle$ ($n > 2N$) of an ensemble with given $\langle b^n \rangle$ ($n \leq 2N$) are nonunique.[3] If $I_{2N}(\lambda) > 0$ (all λ) we can take $\langle b^{2N+1} \rangle = 0$ and verify by minimizing with respect to λ_{N+1} that a sufficient condition for $I_{2N+2}(\lambda) > 0$ (all λ) is

$$\langle b^{2N+2} \rangle > \max[(\sum_{s=0}^{N} \lambda_s \langle b^{s+N+1} \rangle)^2 / I_{2N}(\lambda)], \qquad (4)$$

where the maximum is with respect to the λ_s at $\sum_{s=0}^{N} \lambda_s^2 = 1$. This can be continued indefinitely yielding all $I_{2n}(\lambda) > 0$, provided only that the arbitrarily prescribed $\langle b^{2n} \rangle$ ($n > N$) grow fast enough with n. Very large values of $\langle b^{2N+2} \rangle$ in (4) correspond to making b very large in a fraction of the realizations small enough to contribute negligibly to $\langle b^{2N}\rangle$.

If b in the realizations of an ensemble is constrained to be a root of some polynomial $L = \sum_{s=0}^{N} \lambda_s{}^0 b^s$ then $I_{2N}(\lambda^0)=0$ and the arbitrary extension (4) fails, reflecting a reduction in the freedom of higher moments given the $\langle b^n\rangle$ ($n \leqslant 2N$). Now consider a sequence of values $\langle b^n\rangle$ ($n \leqslant 2N$) which approach finite limits in such a way that $I_{2N}(\lambda)>0$ (all λ) and $I_{2N}(\lambda^0)=\kappa^2$ as $\kappa \to 0$. For every nonzero value of κ the realizability of these moments is verified by the construction above. The limit values are moments of an ensemble which satisfies $\langle L^2\rangle=0$. As $\kappa\to 0$ the right-hand side of (4) goes to infinity; in effect we are demonstrating realizability of the given moments at small κ by showing realizability for a composite ensemble formed by adding to the $\langle L^2\rangle=0$ ensemble a few realizations with very large values of b. The realizations where b is far from a root of L have zero measure in the limit.

These considerations, including the possible construction of higher moments by a generalization of (4), carry over to the case of many variables. The realizability conditions for moments of the y_i are

$$I_{2n}(\lambda) = \langle [\lambda_0 + \sum_i \int_0^T \lambda_i(t) y_i(t)dt + \sum_{ij}\iint_0^T \lambda_{ij}(t,t') y_i(t) y_j(t') dt\,dt' + \text{terms to } n\text{th degree in } y\text{'s}]^2\rangle \geqslant 0, \quad (5)$$

where the λ's are arbitrary real functions satisfying

$$\{\lambda_0{}^2 + \sum_i \int_0^T [\lambda_i(t)]^2 dt + \sum_{ij}\iint_0^T [\lambda_{ij}(t,t')]^2 dt\,dt' + \cdots\} > 0. \quad (6)$$

We can show that a set of values for the many-time moments of the y_i to fourth order represent an ensemble which satisfies (1) in mean squares if and only if they are the limits as $\kappa \to 0$ of values which satisfy (2) and the constraints that

$$I_4(\lambda) > 0 \quad (\text{all } \lambda). \quad (7)$$

In contrast to the single-variable case, (2) constrains the y_i in the limit not to discrete values but to hypersurfaces in the function space.

Although the limit values of the moments in (2) are averages over some ensemble whose members satisfy (1) except for a possible zero-measure set, this does not ensure that all fourth-order moments evolve according to the NS hierarchy equations. To see this, note that the moments $\langle [y_i(t)]^4\rangle$ do not enter (2) because it happens that $A_{ijk}=0$ if two indices are equal. Take a fraction $\sim\kappa^4$ of the realizations and in them take $y_i(t)=0$ except for arbitrary values $O(1/\kappa)$ for $y_1(0)$ and $dy_1(t)/dt$ (all t). The contribution to (2) is $O(\kappa^2)$ but that to $\langle [y_1(t)]^4\rangle$ is $O(1)$. Thus for $\kappa\to 0$ there exist pathological ensembles that satisfy (1) in mean square but have $\langle [y_1(t)]^4\rangle$ varying arbitrarily. They can be weeded out by requiring that there be some finite values of all moments of order $\leqslant 6$ and that $I_6(\lambda)>0$ (all λ). Then it follows from (2) that the NS hierarchy evolution equations for all fourth-order moments are satisfied with $O(\kappa)$ errors. If $I_{2N}(\lambda)>0$ (all λ) and all moments of order $\leqslant 2N$ are finite ($N>2$), then the hierarchy equations for all moments of order $\leqslant (2N-1)$ are satisfied with $O(\kappa)$ errors. If only (7) is imposed, the limit moment values belong to an ensemble such that the NS hierarchy of evolution equations are obeyed with $O(\kappa)$ errors for all fourth-order moments that appear in (2) and for all first-, second-, and third-order moments whether or not they appear in (2).[4]

Initial values of the moments to fourth order are insufficient to determine their later values uniquely because (1) is nonlinear. But conditions on the behavior of these moments throughout the time interval may reduce the ambiguity to zero or unimportance. Thus, consider sationary isotropic turbulence. To maintain it we can add a forcing term $f_i(t)$ to $L_i(t)$, deal with moments of the joint y-f distribution, and make the needed additions to (5). It is reasonable to expect that unique values of moments to fourth order are determined by prescribing the many-time moments of the f_i to fourth order and then imposing a smoothness condition, for example that the normalized variance of the spatially local rate of energy dissipation be minimum.

We now outline two variational treatments of (2). The first is to expand the unknown moments of orders $\leqslant 4$ in some complete sets of orthogonal functions on $0 \leqslant t \leqslant T$. We choose $\rho_i(t)$, admit successively more terms in the expansions, and minimize κ^2 subject to (7) and any specific conditions on the solution. For the stationary turbulence problem the constraint of least dissipation variance could be imposed by minimizing at each stage the sum of $\ln\kappa^2$ and some function of the variance. If the exact solution is unique, we expect convergence to the exact values because the orthogonal functions are complete. If there is not uniqueness, we expect convergence to particular exact values which depend on the choice of orthogonal functions, the $\rho_i(t)$, and the precise way the minimization is carried out. The interval T may have any length. We need not use a semi-infinite interval to treat the steady state. But the choice of T will affect the rate of convergence.

We conjecture that convergence of the approximations can still be obtained if, instead of imposing (7) completely at each stage, the λ functions are also expanded in complete orthogonal functions and successively more terms admitted.

An alternative treatment is to expand the $y_i(t)$ and $f_i(t)$ in Wiener-Hermite (WH) functions[5] of Gaussian white-noise processes in time[6] thereby yielding expansions of the moments in terms of the WH kernels. Successively more of the kernels are admitted and minimization of κ^2 carried out as before. The WH expansion is complete and automatically satisfies (5) for all n at each stage of expansion. We therefore again expect convergence to exact values of the moments. Although the WH expansion of a random process is nonunique,[7] the fact of minimizing κ^2 ensures the optimum expansion here.

The terms in the orthogonal expansions have the same arguments always (n time arguments for nth order moments in the general case). The WH expansions are much more complicated. Intermediate arguments proliferate in the higher orders and the structure closely resembles that of perturbation expansions for the moments. Moreover, the WH kernels are unknown functions which themselves must be expanded for computation. The most difficult part of the orthogonal expansion scheme is (7), which is automatically satisfied in the WH scheme. A combination of Monte Carlo methods and techniques for extraction of minimum eigenvalues of large matrices may be effective in handling (7).

Both of the variational methods can be applied to systems of the general form

$$L_i(t) = dy_i/dt - g_i(y) = 0, \tag{8}$$

where the y_i are real stochastic variables, $g_i(y)$ is a polynomial of degree M in the y's, and there are conservation or other properties that ensure healthy solutions. We simply replace (7) by $I_{2M}(\lambda) > 0$.[8] The methods can also be extended to the case where the $y_i(t)$ are Heisenberg operators and (8) is adjoined to equal-time commutation relations of the form $C_{ij}(y) = 0$, where C_{ij} is of second degree in the y's. Now the y's are not stochastic, but in correspondence to (2) and (7) we consider the limit $\kappa \to 0$ of

$$\sum_i \int_0^T \langle L_i^\dagger(t) L_i(t) \rangle \rho_i(t) dt = \kappa^2, \tag{9}$$

$$\sum_{ij} \int_0^T \langle C_{ij}^\dagger(y) C_{ij}(y) \rangle \rho_{ij} dt = \kappa^2, \tag{10}$$

$$I_{2M}(\lambda) > 0, \tag{11}$$

where $\langle X \rangle$ denotes expectation $\sum_{\alpha\beta} X_{\alpha\beta} \langle \psi_\alpha^* \psi_\beta \rangle$ over a density matrix $\langle \psi_\alpha^* \psi_\beta \rangle$ (α and β label components of the state vector ψ), $\rho_i(t)$ and ρ_{ij} are positive c numbers, daggers denote Hermitian conjugation, and $I_{2M}(\lambda)$ is the expectation of the product of the general Mth degree polynomial in the y's and its Hermitian conjugate. By arguments like those for (7), (11) is a sufficient condition for existence of a positive-definite density matrix such that any set of matrices $[y_i(t)]_{\alpha\beta}$ labeled by i and t have given moment values (expectations of matrix products). Equations (9) and (10) then constrain these moment values and in the limit $\kappa \to 0$ imply that the matrix elements of L_i and C_{ij} between states with nonzero representation in $\langle \psi_\alpha^* \psi_\beta \rangle$ vanish in mean square. The construction (4) for moments satisfying the higher realizability conditions can be taken over by assigning suitably large matrix elements $[y_i(t)]_{\alpha\beta}$ for α and β in the outer reaches of the system configuration space.

WH expansions for the y_i, and expansions of their moments in terms of the WH kernels, can be carried out by using q-number white-noise processes that satisfy commutation relations of the form $[a(t), a^\dagger(t')]_\pm = \delta(t-t')$. As in the c-number case the realizability conditions are automatically satisfied to all orders by the WH expansions.

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[1]R. H. Kraichnan, Phys. Rev. 109, 1407 (1958). We could instead take $\Sigma_i \int_0^T \langle [\overline{L}_i(t)]^2 \rangle \rho_i(t) dt = \kappa^2$, where $\overline{L}_i(t) = \int_0^T L_i(s) ds$. A general form is

$$\Sigma_{ij} \iint_0^T \langle L_i(t) L_j(t') \rangle \rho_{ij}(t, t') dt\, dt' = \kappa^2,$$

where $\rho_{ij}(t, t')$ is positive definite in the sense of being

a possible covariance matrix.

[2]H. S. Wall, *Analytic Theory of Continued Fractions* (Chelsea, New York, 1967), Chap. 17; G. A. Baker, Jr., *Essentials of Padé Approximants* (Academic, New York, 1975), Chap. 17. We admit discrete distributions.

[3]The degree of ambiguity admitted by given moment values can be surprising. By the use of Padé approximants, a discrete b distribution can be constructed such that the moments to eight order have Gaussian values but still b vanishes in 8/15 of the realizations.

[4]There is a subset of (7), defined by restrictions on the λ, that involves only the moments of order ≤ 4 which have $O(\kappa)$ errors and yet provides sufficient conditions for realizability of those moments. Moreover, the full set $I_6(\lambda) > 0$ is not needed to ensure $O(\kappa)$ errors for all fourth-order moments. Those relations that are needed can be expressed as $I_4(\lambda) > 0$ relations for a formally enlarged set of variables. Thus for the example taken one could introduce $q_i(t)$ and require

$$\Sigma_i \int_0^T \langle (q_i - y_i^2)^2 \rangle \rho_i(t) dt = \kappa^2$$

with $\langle q_i^3 \rangle$ finite.

[5]T. Imamura, W. C. Meecham, and A. Siegel, J. Math. Phys. 6, 695 (1965).

[6]WH expansions with basis processes that are not random in time [S. E. Bodner, Phys. Fluids 12, 33 (1969); W. C. Meecham, P. Iyer, and W. C. Clever, Phys. Fluids 18, 1610 (1975)] do not describe sufficiently general steady states of many-time moments.

[7]Bodner, Ref. 6; Meecham, Iyer, and Clever, Ref. 6.

[8]The device of Ref. 4 reduces these relations to the form (7).