A Long-Standing Conjecture and a New Uniqueness Condition for the Solution of the Elastic Unitarity Equation

I. A. Sakmar
Applied Mathematics Department, University of Western Ontario, London, Ontario N6A 3K7, Canada
(Received 31 January 1979)

We improve the sufficient condition for uniqueness of the solution of the elastic unitarity equation for given differential cross section. Uniqueness is proved for $\sin\mu < 0.9577$, which is close to the conjectured condition $\sin\mu < 1$. Here $\sin\mu$ is the supremum of an integral determined by the cross section. Improved conditions depending on the infinum of the integral are also obtained.

It has been almost a decade since a conjecture was made[1] on the uniqueness condition for the solution of the elastic unitarity equation. If we write this equation as

$$\sin\alpha(z) = \frac{q}{2\pi}\int_{-1}^{+1}\int_{-1}^{+1} \frac{|f(x)||f(y)|}{|f(z)|}\cos[\alpha(x)-\alpha(y)]\,\frac{\theta(K)}{\sqrt{K}}\,dx\,dy, \tag{1}$$

where f is the scattering amplitude, the square of which gives the differential cross section, q is the c.m. wave number, $\alpha(z)$ is the unknown phase function, $\theta(K)$ is the step function, and $K = 1 - x^2 - y^2 - z^2 + 2xyz$, the conjecture was that the solution is unique if

$$\sin\mu < 1.$$

Here $\sin\mu$ is the supremum of the integral $\sin\mu(z)$ which is obtained from the Eq. (1) by majorizing $\cos[\alpha(x)-\alpha(y)]$ by 1. I shall also use the infinum of $\sin\mu(z)$ which will be denoted by $\sin\nu$. The bound

$$\sin\mu < 0.79$$

was the best condition given at the time the conjecture was made. Uniqueness conditions depending on $\sin\nu$ were derived by Atkinson, Johnson, and Warnock[2] and more recently by the present author.[3,4]

We obtained recently[5] another uniqueness condition using a different approach. This sufficiency condition is

$$\inf \sin\mu(z) < \sup \sin\gamma(z). \tag{2}$$

Here $\gamma(z)$ is any lower bound for the phase $\alpha(z)$.

In this paper I combine our results of Refs 3–5 to obtain the strongest yet condition for the uniqueness. I have shown in Ref. 3 that

$$\sin\alpha(z) \geq a_n(\mu,\nu)\cos\mu\sin\mu(z), \tag{3}$$

where a_n has a limit for $n\to\infty$ which is given by

$$\lim_{n\to\infty} a_n = (1 - 2\sin\mu\sin\nu + \sin^2\nu)^{-1/2}. \tag{4}$$

Thus

$$\cos\mu\sin\mu(z)(1 - 2\sin\mu\sin\nu + \sin^2\nu)^{-1/2}$$

is a lower bound for $\sin\alpha(z)$. From this we obtain

$$\sup\sin\gamma(z) = \frac{\cos\mu\sin\mu}{(1 - 2\sin\mu\sin\nu + \sin^2\nu)^{1/2}} \tag{5}$$

and when we combine this result with the expression (2) there results

$$\sin\nu < \frac{\cos\mu\sin\mu}{(1 - 2\sin\mu\sin\nu + \sin^2\nu)^{1/2}}. \tag{6}$$

Condition (6) is our result and it is best analyzed with the help of Figs. 1 and 2. Figure 1 shows the right-hand side of (6) as a function of $\sin\nu$ for different values of the parameter $\sin\mu$. The intersection of these curves with the line $u=\sin\nu$ gives three roots which are the solutions of the expression (6) in the case of equality.

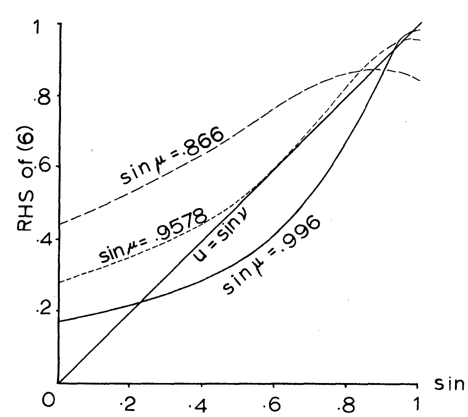

FIG. 1. Parametric solution of Eq. (6): the right-hand side of Eq. (6) as a function of $\sin\nu$ with $\sin\mu$ as a parameter.

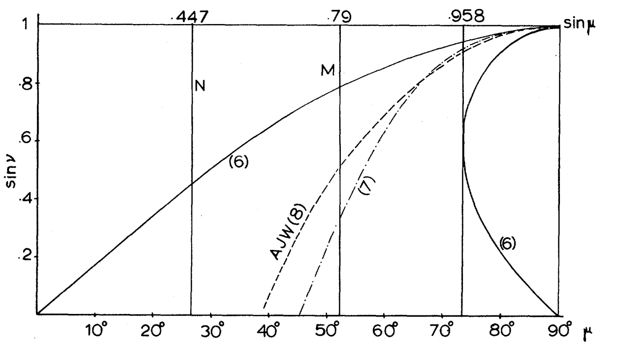

FIG. 2. Plots of all the existing uniqueness conditions: $\sin\nu$ as a function of $\sin\mu$. Vertical lines are $\sin\nu$-independent bounds. N and M refer to Newton's and Martin's bounds (Refs. 1 and 6). To the left of these values, 0.447 and 0.79, the solution is unique. (6), (7), and (8) refer to the equations given in the text. For the curves (7) and (8) the uniqueness regions are the domains above the curves. For the curve (6) the region of uniqueness is the entire domain of the picture except the half-top shaped area at the extreme right between the right branch of the curve (6) and the vertical line at $\mu = 90°$.

It is seen that for

$$\sin\mu < 0.9578$$

the condition (6) is satisfied regardless of the value of $\sin\nu$. This can be read off from Fig. 2 which shows the solution of Eq. (6), namely, $\sin\nu$ as an explicit function of $\sin\mu$. As can be seen, for $\sin\mu > 0.9578$ the curve has three branches. In this region the inequality (6) is still satisfied if $\sin\nu$ is less than the lowest branch or is between the two upper branches. For comparison I have also given my previous result,[3]

$$\sin\nu > \frac{\sin\mu - \cos\mu}{1 - \sin\mu\cos\mu}; \tag{7}$$

the result of Ref. 2,

$$\sin\nu > \sin\mu - \frac{\cos\mu}{(4\tan^2\mu - 1)^{1/2}}; \tag{8}$$

and the results of Ref. 1 and of Newton,[6]

$$\sin\mu < 0.79 \quad \text{and} \quad \sin\mu < 1/\sqrt{5},$$

respectively. Figure 2 shows that except for the region at the extreme right of the figure the conjecture is proven.

This work was supported by the National Research Council of Canada.

[1]A. Martin, Nuovo Cimento A59, 131 (1969).
[2]D. Atkinson, P. W. Johnson, and R. L. Warnock, Commun. Math. Phys. 28, 133 (1972).
[3]I. A. Sakmar, J. Math. Phys. 19, 2124 (1978).
[4]I. A. Sakmar, J. Math. Phys. 20, 11 (1979).
[5]S. K. Chan and I. A. Sakmar, to be published.
[6]R. G. Newton, J. Math. Phys. 9, 2050 (1968).

Radiative-Emission Spectrum of 50-MeV Nuclear Excitations Populated by Proton Capture

L. G. Arnold
Department of Physics, The Ohio State University, Columbus, Ohio 43210
(Received 13 February 1979)

Qualitative features of the γ-emission spectrum from nuclear states in the 50-MeV excitation energy range to states in the giant-dipole-resonance region are described. Properties of the initial- and final-state structures are deduced.

In an exploratory study of proton radiative-capture reactions at the Indiana University Cyclotron Facility, Kovash *et al.*[1] observed intense primary radiation to excitations of the residual nucleus which were identified as stretched-configuration $\hbar\omega_0$ particle-hole states in closed-shell nuclei and the corresponding single-particle states in closed-shell-plus-one nuclei. The purpose of this Letter is to describe qualitative features of this radiation and the structure of the excitations involved. These features indicate that proton radiative-capture reactions for incident protons with energy greater than about 25 MeV may be used to map out the density of one-body states from the Fermi level to 60 MeV excitation, perhaps higher.

A schematic representation of the radiation observed[1] in closed-shell residual nuclei is shown