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Position-Space Renormalization-Group Transformations: Some Proofs and Some Problems

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It is shown that certain types of position-space, or "cell," renormalization-group transformations from an Ising-model object Hamiltonian onto another Ising-model image Hamiltonian are well defined and smooth in the thermodynamic limit, provided the object Hamiltonian has a large enough magnetic field. In certain other cases there is evidence (though not a rigorous proof) that no thermodynamic limit exists, or, if it exists, the transformation is not smooth.

Renormalization-group methods have been applied with considerable success to a variety of problems in statistical mechanics, especially in connection with phase transitions and critical points.[1] Despite their widespread use, little is known about the precise mathematical structure of renormalization transformations: In particular, whether they possess a well-defined thermodynamic limit as smooth transformations from Hamiltonians onto Hamiltonians, as is assumed (at least implicitly) in practical applications.

We have been able to prove that a certain class of position-space or "cell"-type transformations for Ising models are well defined and smooth in this limit provided the starting or "object" Hamiltonian has a large enough magnetic field; i.e., the lattice-gas activity is sufficiently small. In certain other cases we have plausible, though not completely rigorous, arguments for the existence of "peculiarities" in certain transformations, which suggest that either the transformation is not smooth or the "image" Hamiltonian (supposedly) produced by the transformation is not defined. In those regions of the parameter space in which they occur these peculiarities, which do *not* seem to have any direct connection with the phase transitions associated with the object Hamiltonian, cast doubt on the applicability of the usual renormalization-group procedures.

The transformations we have studied are of the form[2,3]

$$\exp H'(\tau) = \mathrm{Tr}_\sigma[T(\tau,\sigma)\exp H(\sigma)], \tag{1}$$

where σ stands for a set of Ising spins σ_i which comprise the object system with "Hamiltonian" (equal to the usual Hamiltonian divided by $-kT$) $H(\sigma)$, and τ another set of Ising spins τ_j which comprise the image system with Hamiltonian $H'(\tau)$. The trace Tr_σ denotes a sum over the values, ± 1, of all of the σ's. The conditional probability T is of the form introduced by Kadanoff[3]:

$$T(\tau,\sigma) = \prod_j [2\cosh(p\sum_{i\in C(j)}\sigma_i)]^{-1} \times \exp\{p\tau_j \sum_{i\in C(j)}\sigma_i\}. \tag{2}$$

Here $C(j)$ stands for the set of sites of the object system lying in a "cell" associated with the image spin τ_j, and p is a parameter giving the coupling of the image spin to the object spins in its cell.

The transformation (1) is well defined for a finite system in which the σ's are associated with a finite subset Ω of sites taken from an infinite lattice, and the τ's with a finite subset Ω' of another infinite lattice. In order to discuss the thermodynamic limit in which both Ω and Ω' increase to infinite size, we introduce the function $W'(A|X)$ which is the *change* in $H'(\tau)$ in going from a configuration in which the τ_i are $+1$ for i in the set X and -1 elsewhere to a configuration in which the τ_i are $+1$ for i in the set $X \cup A$ and -1 elsewhere. In lattice-gas language it is the change in energy (divided by $-kT$) which occurs if particles are added at the empty sites in A when all the sites in X are already occupied. This function can be finite if A is finite, which we shall always assume to be the case, even if the set X is infinite, and thus retains its significance (unlike the total energy) for an infinite system. In particular it can be shown[4] that $W'(A|X)$ corresponds to a well-defined H', or set of interactions, for the infinite system provided it is a continuous[5] function of its second argument.

The function $W_{\Omega}'(A|X)$ is defined in the same manner for a finite image system Ω', assuming $A \subset \Omega'$, except that X in the above discussion must be replaced by $X \cap \Omega'$. By combining (1) and (2) one obtains, if $A \cap X$ is empty, the expression

$$\exp W_{\Omega}'(A|X) = \langle \exp(2p \sum_{i \in C(A)} \sigma_i)\rangle_{\Omega, X}, \tag{3}$$

where the angular brackets refer to an average defined for a *modified* object system by

$$\langle \Theta \rangle_{\Omega, X} = \frac{\mathrm{Tr}_{\sigma}[\Theta(\sigma) \exp H_X(\sigma)]}{\mathrm{Tr}_{\sigma}[\exp H_X(\sigma)]}. \tag{4}$$

Here H_X is the modified-object-system Hamiltonian defined by

$$H_X(\sigma) = H(\sigma) + p \sum_{i \in C(X \cap \Omega')} \sigma_i - p \sum_{i \in C(\Omega' \setminus X)} \sigma_i + \sum_{j \in \Omega'} \sum_{D \subset C(j)} Q_D \sigma_D, \tag{5}$$

with

$$C(A) = \bigcup_{j \in A} C(j), \quad \sigma_D = \prod_{i \in D} \sigma_i, \tag{6}$$

and $\Omega' \setminus X$ the complement of X in Ω'. The final term in (5) comes from the terms in square brackets in (2), and the Q_D are real-valued functions of p.

It can be shown that the thermodynamic limit for the image-system interactions, H', is well defined if, for every (finite) set A and every (in general infinite) set X which does not intersect A, the right-hand side of (3) possesses a limit as Ω (together with Ω') tends to infinity, *provided* this limit is uniform. By "well defined" we mean that $W'(A|X)$, the limit of $W_{\Omega}'(A|X)$, is a continuous function of X and that it is associated with the Gibbs state, which is the thermodynamic limit of the probability distribution

$$\rho'(\tau) = \exp H'(\tau)/\mathrm{Tr}_{\tau}[\exp H'(\tau)], \tag{7}$$

through the equilibrium equations of Dobrushin[6] and Lanford and Ruelle.[7,8] The proof of these assentions is somewhat technical and will be published elsewhere.[4]

We have used the equation of Gallavotti and Miracle-Sole[9] to show that, provided the interactions $H(\sigma)$ translated into lattice-gas language[10] satisfy the fairly mild restriction[11]

$$\|\Phi\| = \sup_i \sum_{X: i \in X} |\Phi(X)| < \infty, \tag{8}$$

and *provided* the magnetic field is sufficiently large (i.e., the lattice-gas activity is sufficiently small), the thermodynamic limit of (3) is uniform and the resulting $W'(A|X)$, and hence the corresponding interactions in H', are analytic functions of the parameters which appear in H. Under these same conditions, the lattice-gas interaction Φ' corresponding to H' has finite norm in the sense of (8). Moreover, if H is suitably short ranged, the many-body interactions in H' associated with a cluster of sites fall off exponentially rapidly with the size of the cluster.

Unfortunately, our arguments only work when p is finite, while (2) remains well defined in the limit of p going to infinity. Also we are unable to show that the transformation (1) is well defined when applied to a Hamiltonian resulting from a previous application of the transformation. That is to say, we are unable to show that (1) can be iterated more than once.

Our failure to construct a proof does not, of course, imply that (1) fails to possess a thermodynamic limit which is a smooth transformation. However, we also have positive evidence for a breakdown in smoothness or the nonexistence of a thermodynamic limit under some circumstances. Consider the Hamiltonian

$$H(\sigma) = K \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i, \tag{9}$$

with $K > 0$ (ferromagnetic) and the first sum over pairs of nearest-neighbor sites on a lattice of dimension $d \geq 2$. For $T(\tau, \sigma)$ choose the special case ("model I") in which there is one image spin for each object spin; i.e., $C(j)=\{j\}$ is one site. Then the last term in (5) is a constant and can be ignored.

If X in (3) is the empty set $\emptyset$, the modified object Hamiltonian (5) is the same as (9) except that h is replaced by $h-p$. In the case where A is a single site j, the right-hand side of (3) is equal to

$$\cosh 2p + \langle\sigma_j\rangle_{\Omega,\emptyset} \sinh 2p. \qquad (10)$$

It is well known[12] that when K is sufficiently large, (9) leads to a phase transition with the consequence that $\langle\sigma_j\rangle$ in the thermodynamic limit is a discontinuous function of h at $h=0$. The same transition occurs for the modified object system with Hamiltonian $H_{X=\emptyset}$ except that the discontinuity occurs at $h=p$. Thus the thermodynamic limit of (10), and hence $W'(\{j\}|\emptyset)$, is discontinuous at $h=p$. Note that in this case, however, we have not proved that the W' resulting from the thermodynamic limit of (3) is associated with the corresponding Gibbs state through the equilibrium equations.

By considering the nature of the modified object Hamiltonian for various choices of the set X one can make a plausible, though not a rigorous, argument that the function W' is afflicted with similar peculiarities for all values of h between $-p$ and p, provided K is sufficiently large. Analogous arguments—plausible, but not rigorous—suggest the presence of peculiarities for suitable choices of $H(\sigma)$ for other cases of transformation described by (2), both when p is finite and when it is infinite.

Though there can be little doubt as to the existence of these peculiarities under appropriate conditions, their interpretation is difficult. We are of the opinion that the discontinuities which occur in the thermodynamic limit of (3) for model I, and in certain other cases, are indications that the transformation (1) does not possess a well-defined thermodynamic limit, that is to say, an appropriate image Hamiltonian does not exist. If this is correct, it is rather disturbing in view of the fact that for $h=p$ one is quite certain[13] that (9) does *not* give rise to a phase transition in the object system. In certain other cases we have examined, it seems possible that the thermdynamic limit exists, but that the parameters in H' are not smooth functions of those in H. From the point of view of typical applications, an unsmooth transformation is almost as disturbing as a nonexistent transformation. Further study is needed to determine the precise nature of these peculiarities and the extent to which (if any) they invalidate the results of approximate position-space renormalization-group calculations.

All the peculiarities we have discovered thus far arise from phase transitions in a *modified* object system which seem to have no connection at all with phase transitions in the actual object system for the same parameter values. This suggests—though such a suggestion must be regarded as quite speculative—that the peculiarities may arise from taking (1) "literally," and that a modified H' which closely reproduces the probabilities of the more likely configurations and changes those of the less likely configurations, and thus has the "right physics," could be produced by an approximate transformation lacking the pathologies discussed above. Indeed, the approximations which are actually used to calculate properties of Ising models may already embody this suggestion. If so, one can understand their considerable success as approximations, together with their apparent failure to provide instances of the peculiarities which are (almost certainly) present in exact transformations.

Our results do not apply directly to transformations on non-Ising systems and those which involve integrating out degrees of freedom in momentum space. It is evident, however, that there is a need for a careful investigation of the mathematical properties of such transformations to determine the extent to which they are well defined and smooth in the thermodynamic limit.

We are sometimes asked whether it is not possible to avoid problems associated with the thermodynamic limit by regarding the renormalization transformations simply as mappings of states (i.e., probability distributions) of infinite systems onto other states, without raising the question as to whether these states are associated with a Hamiltonian. Such transformations are easier to define and are probably better behaved than transformations of Hamiltonians onto Hamiltonians.

Our reply is that whereas such an approach may have distinct advantages, it is not clear (to us, at least) how the standard renormalization-group phenomenology of smooth flows, relevant eigenoperators, etc., can be translated into the space of states. The usual discussions involve properties of transformations near, and not simply at, the fixed point. No doubt if there were a relatively simple, straightforward connection between states and Hamiltonians, there would be no great difficulty in translating a phenomenology for one into a phenomenology for the other. But the results reported in this paper, together with the enormous literature on phase transitions, suggest that the connection between states and Hamiltonians is extremely complex.

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Time Invariance of Planck's Constant

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We examine the effect of an assumed variation in Planck's constant $\hbar$ on the element abundances produced by nucleosynthesis in the standard hot big-bang model of the universe. In order to be consistent with current estimates of the primordial helium and deuterium abundances, we find that at the epoch of nucleosynthesis (a red shift $z \approx 10^8-10^{10}$, $t \approx 10^{-1}-10^3$ sec after the initial event), $0.2 \lesssim \hbar_{\text{nucleosynthesis}}/\hbar_{\text{present}} \lesssim 3$. This result supports the applicability of the local laws of physics to the earliest epoch of the universe examined to date.

Ever since Dirac[1] first examined the possibility of a variation with cosmic time of the fundamental "constants" of nature, the applicability of the local laws of physics to other places and times in the universe has been subject to observational scrutiny. Two points of view have been taken: (1) that the laws of physics (including the physical "constants") evolved at a very early stage of the universe and were subsequently "frozen in" with their present values,[2] or (2) that the laws have evolved continuously in time, the point of view proposed by Dirac and the one examined in this paper.

Direct and indirect arguments based on astronomical observations of distant objects or of radioactive nuclei (formed at an earlier epoch) in the laboratory have been used to set limits on any allowed time variation of the atomic constants such as the fine-structure constant, the photon-to-electron inertial-mass ratio, the anomalous magnetic moment of the proton, the charge on the electron or proton, and the weak-interaction constant. These results have been summarized by Dyson.[3] By the assumption that the atomic constants have all remained fixed, limits on any possible time variation of Newton's gravitation constant G have also been set.[4]

Recently, observational limits on variations of