with a reduced effective frequency ω'.

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Effects of Voids on the Thermal Magnetoresistivity of Metals

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The influence of cylindrical voids on the thermal magnetoresistance of a model metal is calculated. In the absence of lattice thermal conduction a linear thermal magnetoresistivity results from the presence of the voids in a manner similar to the electrical magnetoresistivity. However, when the lattice conductivity is present, marked deviations from linearity occur.

The question of the anomalous magnetoresistances of the simple metals is considered to be one of the great unsolved problems of metals physics. After nearly 50 years of research, the source of these anomalies is still unknown. The semiclassical magnetoresistance theory of Lifshitz, Azbel', and Kagnaov[1] predicts that closed-orbit, uncompensated metals should have electrical and thermal magnetoresistivities which saturate in strong fields. However, in many of the simple metals[2-4] (the alkalis[2] and metals such as indium[3] and aluminum[4]) a linear transverse electrical magnetoresistivity is observed. In potassium, the archetypical simple metal, the transverse thermal magnetoresistance contains terms linear and quadratic in the field[5] (similarly, the other magnetotransport coefficients show unexplained behavior). In fact, to our knowledge, no experiments in the simple metals have been reported in which a saturating magnetoresistivity has been observed. The question of whether the source of the anomalies is intrinsic or extrinsic has not been resolved. Noting that the linear magnetoresistance of the simple metals varies unpredictably with sample handling and fabrication techniques, several authors[6,7] suggested that sample inhomogeneities (voids, inclusions, grain boundaries, dislocations, etc.) may be responsible for the anomalous behavior. Fletcher has also argued in a different manner for an extrinsic cause.[8] A recent Letter by Beers *et al.*[9] describes a measurement of the electrical magnetoresistance of a pure indium specimen into which cylindrical voids were introduced. Beers *et al.* find a large enhancement in the linear electrical magnetoresistance due to the presence of the voids.

In this Letter we report a calculation of the thermal magnetoresistance of a model metal containing cylindrical voids, including explicitly the effects of lattice thermal conductivity. Our results demonstrate that the presence of voids does *not* enhance the quadratic term[10] and that a nonzero lattice thermal conductivity substantially *reduces* the effects of the voids in creating a linear magnetoresistivity, producing, also, a marked deviation from linearity at high fields. Thus, the presence of the lattice conductivity results in large differences in the effects of voids on the electrical and thermal magnetoresistivities. This result can be used to determine the role of sample inhomogeneities in the magnetoresistance anomalies of the simple metals.

A number of theories have been proposed relating the linear electrical magnetoresistivity to various types of sample inhomogeneities. Our calculations are similar to those of Sampsell and Garland[6] and of Stroud and Pan.[7] They calculate the effective electrical magnetoresistance of a free-electron metal within which are large-scale voids ($d>l$, where d is a typical void dimension and l is the electron mean free path). The effects of a variety of shapes of voids are calculable, but, for simple comparison with the work of Sampsell and Garland[6] and of Beers *et al.*,[9] in this Letter we only discuss cylindrical voids whose axes lie perpendicular to $\vec{H}$ and $\vec{J}_Q$, the thermal current. In our calculations we have used both the boundary-value approach of Sampsell and Garland[6] and the effective-medium approach of Stroud and Pan.[7] We present here the boundary-value approach because we feel this method displayes the physics in an obvious manner.

For the calculation, we make the basic assumption that the metal is described by a free-electron model and, furthermore, that there is a field-independent lattice conductivity present. We also assume that the electrons are in the hydrodynamic limit and that the temperature is sufficiently low that radiation effects may be neglected. This ensures the same boundary conditions for the electrical and thermal currents and potentials.

Since there are no sources of heat in the specimen, we find that in the presence of a magnetic field the temperature satisfies the equation

$$\nabla_i \kappa^{ij} \nabla_j T = 0. \tag{1}$$

This equation obtains for inhomogeneous and anisotropic conductivity tensors.

For the model described above, the conductivity tensor has the form

$$\kappa^{ij} = \kappa_0 \begin{pmatrix} \gamma+\delta & \beta\gamma & 0 \\ -\beta\gamma & \gamma+\delta & 0 \\ 0 & 0 & 1+\delta \end{pmatrix}, \tag{2}$$

where κ_0 is the free-electron thermal conductivity in zero field, δ is the ratio of the lattice thermal conductivity to the free-electron thermal conductivity, $\beta = \omega_c \tau_{\rm th}$, $\gamma = (1+\beta^2)^{-1}$, ω_c is the cyclotron frequency, and $\tau_{\rm th}$ is the mean time between collisions catastrophic to thermal conductivity. While T does not satisfy Laplace's equation in the original coordinate system, it is always possible to find a coordinate system in which it does. For the problem at hand, elliptic cylindrical coordinates do the job. The temperature distribution, consistent with the boundary conditions that the thermal current is injected uniformly at a great distance from the void and perpendicular to the field, takes the form

$$T = -(J_Q{}^0/\kappa_0)\{A \cosh\mu \cos\theta + By + C \cos\theta \exp[-(\mu-\mu_0)]\}. \tag{3}$$

In this equation,

$$A = R_0 \frac{(1+\rho)(1+\beta^2)}{(1+\rho)^2+\beta^2} \left(\frac{1-\gamma}{1+\delta}\right)^{1/2}, \tag{4}$$

$$B = \frac{\beta(1+\beta^2)}{(1+\rho)^2+\beta^2}, \tag{5}$$

$$C = R_0[\gamma(1+\delta)(1+\rho)]^{-1/2}, \tag{6}$$

$$\rho = \delta\gamma^{-1}, \tag{7}$$

$$\sinh\mu_0 = \left(\frac{\gamma+\delta}{1-\gamma}\right)^{1/2}, \tag{8}$$

and R_0 is the radius of the void, μ and θ are the elliptic cylindrical coordinates, and $J_Q{}^0$ is the limiting value of the injected current. This yields the following current:

$$J_Q{}^x = J_Q{}^0\left\{1 - \exp[-(\mu-\mu_0)]\left(\frac{\gamma+\delta}{1-\gamma}\right)^{1/2} XY\right\}, \tag{9}$$

$$J_Q{}^y = J_Q{}^0 \exp[-(\mu-\mu_0)]\left(\frac{\gamma}{\gamma+\delta}\right)^{1/2} XY, \tag{10}$$

$$J_Q{}^z = -J_Q{}^0 \exp(\mu_0)\left(\frac{1+\delta}{1-\gamma}\right)^{1/2} Y \sin\theta \cos\theta, \tag{11}$$

where

$$X = \sinh^2\mu \cos^2\theta - \cosh^2\mu \sin^2\theta \tag{12}$$

and

$$Y^{-1} = \sinh^2\mu \cos^2\theta + \cosh^2\mu \sin^2\theta. \tag{13}$$

In the limit $\delta \to 0$, the foregoing expressions reduce to those of Sampsell and Garland. We should also mention that the formalism described above can deal effectively with problems considerably more complicated than the one discussed here. Equation (1), combined with a choice of κ^{ij}, displays the physics of the model in as perspicuous a manner as possible. We also point out that if κ^{ij} does not depend on the coordinates, i.e., $\kappa^{ij}\nabla_i\nabla_j T = 0$, then T is determined solely by the symmetric part of κ^{ij}. Antisymmetric modifications of κ^{ij} do not affect the temperature distribution. The symmetric part of κ^{ij} may always be diagonalized, and a rescaling of the coordinates reduces the equation for the temperature to Laplace's equation. This was the result used above.

Once the thermal currents and fields have been obtained, it is a straightforward matter to get the effective thermal conductivity. This is obtained by numerically integrating the entropy production over the volume of the specimen, or, directly, by using the method of Stroud and Pan. Some insight into the physics may be gained from Fig. 1, a plot of the entropy production as a function of the horizontal (x) distance from the void at a height $z = R_0$. We show the results for $\beta = \omega_c \tau_{\text{th}} = 100$, for various values of δ. For $\delta = 0$ the results are identical to those obtained by Sampsell and Garland (Ref. 6, Fig. 3), these results have been normalized to unit entropy production far from the void. The interpretation of the $\delta = 0$ curve is given by Sampsell and Garland; the large peaks in the "dissipation" in the void "shadow" result from the large current sheets which form there when $\omega_c\tau_{\text{th}} > 1$. The effect of a nonzero δ is twofold: The asymptotic limit of the entropy production increases (at large β and μ, $\kappa^{xx} \to \delta\kappa_0$ and the fields and currents are uniform) and the height of the peaks is diminished. For sufficiently large δ or β, the peaks "disappear." The conductivity tensor has become field independent and the zero-field results obtain, albeit with a much smaller conductivity.

Since most of the entropy production involved in the linear magnetoresistance comes from the two peaks in the void shadow, we expect substantial deviations from linearity to occur at high β. This is indeed the case, as is shown in Fig. 2, a plot of $[\Delta W/fW_0]_{\text{extra}}$ vs $\omega_c\tau_{\text{th}}$ for various δ. Here

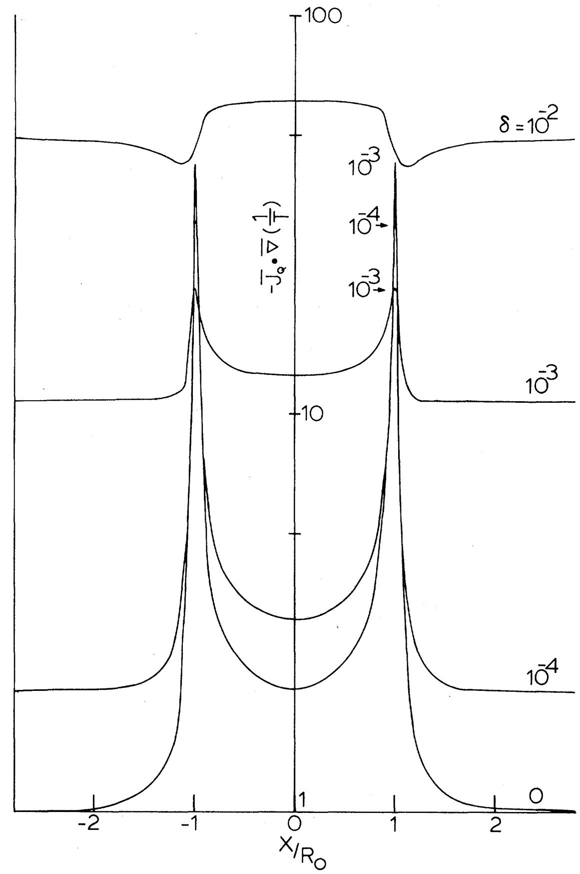

FIG. 1. Volume entropy production $-\vec{J}_Q \cdot \nabla(T^{-1})$, plotted along a line parallel to the x axis which touches the cylinder at $Z = R_0$. The curves are for various values of δ, the relative lattice thermal conductivity and are all for $\omega_c\tau_{\text{th}} = 100$. The peak values are indicated near the right-hand lobe.

we have

$$\left.\frac{\Delta W}{f W_0}\right|_{\text{extra}} = \frac{W_{\text{calc}}(H) - W_{\text{no void}}^{(H)}}{W(H=0)F}, \tag{14}$$

where $W_{\text{calc}}(H)$ is the effective thermal resistivity as determined from a numerical calculation or directly from the effective-medium theory of Stroud and Pan. $W_{\text{no void}}^{(H)}$ is the thermal resistance of the system without the voids, and f is the volume fraction of voids. Thus, only the resistivity due to the presence of the voids will appear in Fig. 2. For $\delta = 0$, we again reproduce the results of Sampsell and Garland, that is, $[\Delta W/fW_0]_{\text{extra}}$ is linear in $\omega_c\tau_{\text{th}}$, with a Kohler slope $1.00f$, for $\omega_c\tau_{\text{th}} > 10$. However, for nonzero δ the results show marked deviations from linearity. For the various δ's, the deviation begins at that field at which the lattice conductivity becomes

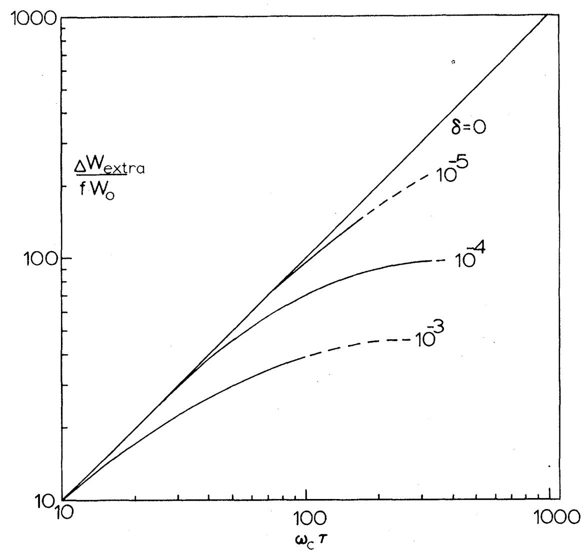

FIG. 2. The "extra" thermal resistivity (see text) vs $\omega_c\tau_{th}$ for various δ.

an appreciable fraction of the total conductivity, i.e., when δ is a significant fraction of γ. We present results for $\omega_c\tau_{th}$ up to 300; this covers the range of $\omega_c\tau_{th}$ readily accessible in many of the alkali metals. As $\omega_c\tau_{th}\to\infty$, the curves flatten with $[\Delta W/f W_0]_{extra} \propto \delta^{-1}$.

There are very little high-field limit magnetoresistivity data available to check these predictions. Potassium, as mentioned above, has a transverse thermal magnetoresistivity which contains terms linear and quadratic in field. The linear term appears to persist to values of $\omega_c\tau_{th}$ of nearly 350. Since δ for potassium is at least 3×10^{-4}, deviations from linearity should be seen easily. However, there is currently considerable controversy concerning the origin of the quadratic term and the magnitude of the lattice conductivity in potassium,[11,12] the data are not conclusive.

Finally, we see that in contrast to the electrical case, the introduction of large voids into a metal does not result in a linear term in the thermal magnetoresistance, at least in strong fields. Clearly these predictions allow for a very positive experimental test of the void hypothesis; we are planning such an experiment.

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