ized, with a current $I > I_A$, can (i) propagate with an equilibrium determined by its self-fields, as predicted by Yoshikawa,[5] and (ii) set up a reversed-field plasma configuration by inducing currents in the plasma and wall of a closed, initially field-free, metal tube.

In the present experiments, the configuration resembles a linear reversed-field pinch. It is possible to envisage extensions of this technique to produce plasma configurations with closed field lines. These could be further heated by the injection of intense neutral, electron, or ion beams; or by an imploding liquid metal liner, as in the Naval Research Laboratory LINUS fusion concept.[8]

[1]C. A. Kapetanakso, W. M. Black, and C. D. Striffler, Appl. Phys. Lett. 26, 368 (1975).
[2]D. A. Hammer, A. E. Robson, K. A. Gerber, and J. D. Sethian, Phys. Lett. 60A, 31 (1977).
[3]C. W. Roberson, D. Tzach, and N. Rostoker, Appl. Phys. Lett. 32, 241 (1978).
[4]J. D. Sethian, D. A. Hammer, K. A. Gerber, D. N. Spector, A. E. Robson, and G. C. Goldenbaum, Phys. Fluids 21, 1227 (1978).
[5]S. Yoshikawa, Phys. Rev. Lett. 26, 295 (1971).
[6]G. Schmidt, Phys. Fluids 5, 994 (1962).
[7]J. D. Sethian, Naval Research Laboratory Memorandum Report No. 3785 (unpublished).
[8]D. L. Book, A. L. Cooper, R. Ford, K. A. Gerber, D. A. Hammer, D. J. Jenkins, A. E. Robson, and P. J. Turchi, in *Proceedings of the Sixth International Conference on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden, West Germany, 1976* (International Atomic Energy Agency, Vienna, 1976), Vol. III, p. 507.

Stability of an Anisotropic High-β Tokamak to Ballooning Modes

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We have applied the Kruskal-Oberman energy principle to a simple model of an anisotropic tokamak in which the pressure varies around flux surfaces. We show that the weighting of pressure towards regions of favorable curvature leads to a significant stabilization of the high-n ballooning modes.

Following recent theoretical investigations of the MHD (magnetohydrodynamic) stability of *scalar*-pressure tokamaks,[1-6] it is now generally believed that the upper limit to β is set by the ballooning mode. Apart from its use as an additional heat source, neutral injection has been proposed as a method for "pumping-up" β in the flux-conserving tokamak[7]; it is also fundamental to the counterstreaming ion concept.[8] These applications have led us to consider the MHD stability of an anisotropic model of tokamak to high-n ballooning, n being the toroidal mode number.

Our analysis is based on the Kruskal-Oberman energy principle[9]; using the property of adiabatic invariance, Andreoletti[10] has shown their result to be independent of the form of distribution function. We assume that neutral injection is applied at an angle to the magnetic field such that hot ions are created only in the untrapped region of velocity space, so that the distribution function for the trapped particles is not significantly anisotropic. Then for small inverse aspect ratio, δ, the kinetic term in Kruskal-Oberman is $O(\delta^{7/2})$,[11] whereas the fluid terms are $O(\delta^2)$, when $\beta\sim\delta$. Thus, we drop the kinetic term, anticipating that our general analysis will be applied to a large–aspect-ratio model. Writing the fluid terms in a form as closely analogous to that for scalar pressure[5] as possible, we obtain

$$\delta W = \int d\tau \left\{(1-\sigma_-)\vec{Q}_\perp{}^2 - (1-\sigma_-)\frac{\vec{J}\cdot\vec{B}}{B^2}(\vec{\xi}\times\vec{B}\cdot\vec{Q}) - 2(\vec{\xi}\cdot\vec{\kappa})(\vec{\xi}\cdot\nabla\bar{p}) + B^2(1+\sigma_\perp)\left[\left(1+\frac{1-\sigma_-}{1+\sigma_\perp}\right)\vec{\xi}\cdot\vec{\kappa}+\nabla\cdot\vec{\xi}\right]^2 + B^2\left(\frac{1-\sigma_-}{1+\sigma_\perp}\right)(\sigma_\perp+\sigma_-)(\vec{\xi}\cdot\vec{\kappa})^2\right\}, \tag{1}$$

where $\bar{p}=(p_\perp+p_\parallel)/2$, $\sigma_-=(p_\parallel-p_\perp)/B^2$, $\vec{Q}=\mathrm{curl}(\vec{\xi}\times\vec{B})$, $\xi_\parallel\equiv 0$, and $\vec{\kappa}$ denotes the field-line curvature. In order to define $\sigma_\perp$, we introduce the pressurelike moment

$$C=\sum_j m_j\iint\frac{B\,d\mu\,d\epsilon}{v_\parallel}\frac{\partial f_j}{\partial\epsilon}(\mu B)^2, \tag{2}$$

where the velocity-space variables are $\epsilon = \frac{1}{2}v^2$ and $\mu = \frac{1}{2}v_\perp^2/B$, so that $v_\parallel^2 = 2(\epsilon - \mu B)$. Thus $\sigma_\perp = (2p_\perp + C)/B^2$. It follows from Eq. (2) and the definitions of $p_\perp$ and $p_\parallel$ that C and $\bar{p}$ are related by[12]

$$\vec{B}\cdot\nabla\bar{p} = \frac{2\bar{p}+C}{2B^2}\vec{B}\cdot\nabla(\tfrac{1}{2}B^2). \tag{3}$$

In practice, the criteria for stability to the "firehose" and "mirror" modes,[13,14] namely, $1-\sigma_- > 0$ and $1+\sigma_\perp > 0$, will be satisfied.

Following Dobrott *et al.*[5] and expanding in $1/n$, we find that to lowest order the minimizing displacements satisfy $\nabla\cdot\vec{\xi} = O(1)$ and $\vec{B}\cdot\nabla\vec{\xi} = O(1)$. The lowest-order contribution of the kink term [second in Eq. (1)] vanishes, and after minimization with respect to the first-order displacement, $\vec{\xi}^{(1)}$, the "field compression" term (fourth) also vanishes. In zeroth order, δW is then a functional of the zeroth-order $\vec{\xi}$ only. Employing the usual axisymmetric (ψ, χ, φ) coordinate system, we express $\vec{\xi}$ as a Fourier mode $\vec{\xi} = \vec{X}(\psi,\chi)e^{in\varphi}$, and obtain

$$\delta W^{(0)} = \int d\tau\left\{(1-\sigma_-)\left[\frac{|\vec{B}\cdot\nabla\xi_\psi|^2}{|\nabla\psi|^2} + \frac{|\nabla\psi|^2}{B^2}\left|\frac{1}{n}\frac{\partial}{\partial\psi}(\vec{B}\cdot\nabla\xi_\psi)\right|^2\right] - \left[(\xi_\psi\kappa_\psi + \xi_s\kappa_s)\left(\xi_\psi{}^*\bar{p}_\psi + \frac{|\nabla\psi|}{B}\xi_s{}^*\bar{p}_s\right) + \text{c.c.}\right] + \frac{B^2(1-\sigma_-)(\sigma_\perp+\sigma_-)}{(1+\sigma_\perp)}|\xi_\psi\kappa_\psi + \xi_s\kappa_s|^2\right\}, \tag{4}$$

where

$$\vec{\xi} = \frac{\nabla\psi}{|\nabla\psi|^2}\xi_\psi + \frac{\vec{B}\times\nabla\psi}{B^2}\xi_s$$

with

$$\xi_s = \frac{1}{in}\frac{\partial\xi_\psi}{\partial\psi};\quad \vec{\kappa} = (\nabla\psi)\kappa_\psi + \frac{\vec{B}\times\nabla\psi}{|\nabla\psi|^2}\kappa_s;\quad \bar{p}_\psi = \frac{\partial\bar{p}}{\partial\psi};\quad \text{and } \bar{p}_s = \frac{\vec{B}\times\nabla\psi\cdot\nabla\bar{p}}{B|\nabla\psi|}.$$

Minimizing Eq. (4) with respect to ξ_ψ, we obtain an Euler equation containing partial derivatives which act on the rapid ψ variation of ξ_ψ, as well as derivatives with respect to χ; following Connor, Hastie, and Taylor,[6] this equation is reduced to an ordinary differential equation. Thus, we define the transformation $\xi_\psi(\psi,\chi) \to F(\psi, y)$ by

$$\xi_\psi = \sum_m e^{im\chi}\int_{-\infty}^{\infty} F(\psi, y)\exp[-i(my + n\int^y \nu d\chi')]dy,$$

where $\nu = B_\varphi/|\nabla\psi||\nabla\chi|$, and all the rapid ψ variation of ξ_ψ is contained within the phase factor $\exp(-in \times \int^y \nu d\chi')$. If we define $G(\psi, y) = \int_0^y \nu d\chi$, then in transform space the Euler equation becomes

$$\frac{1}{J}\frac{\partial}{\partial y}\left[\frac{(1-\sigma_-)}{|\nabla\psi|^2}\left\{1 + \frac{|\nabla\psi|^4}{B^2}\left(\frac{\partial G}{\partial\psi}\right)^2\right\}\frac{1}{J}\frac{\partial F}{\partial y}\right] + 2\left(\bar{p}_\psi - \frac{\kappa_\psi}{\kappa_s}\frac{|\nabla\psi|}{B}\bar{p}_s\right)\left(\kappa_\psi - \kappa_s\frac{\partial G}{\partial\psi}\right)F = 0, \tag{5}$$

where J is the Jacobian of the (ψ, χ, φ) coordinate system. From the mode radial structure defined above, we deduce the physical boundary condition $|y|^{1/2}F \to 0$ as $|y| \to \infty$. Asymptotic analysis of Eq. (5) leads to the localized interchange criterion,[12] as was noted in the scalar-pressure case.[6] In general, anisotropic equilibria are of the form $\bar{p} = \bar{p}(\psi, \chi)$; this suggests that if equilibria can be produced such that the pressure surfaces are displaced inwards relative to the flux surfaces, then the "loading" of pressure into regions of favorable curvature could lead to stability at higher β. We now demonstrate this to be the case.

In coordinates (r, θ) based on the plasma center, the major radius is $R = R_0 + r\cos\theta$. We expand the equilibrium equations[13] in δ and choose the form $\bar{p} = P(\psi)[1 + \alpha(r/a)\cos\theta] + O(\delta^2)$, where P is $O(\delta)$, α is constant, and a is a scale length. Then consistent with this choice the leading-order axial current density is $j_\varphi = -g'(\psi) - xP'(\psi)(2 + \alpha x/a)$, where $x = r\cos\theta$. Choosing linear $g(\psi)$ and $P(\psi)$, and setting $\psi = 0$ on the circular boundary of radius a, we obtain

$$\psi(r,\theta) = \frac{-B_0a^2}{\bar{q}}\left(\frac{1}{2(1+\frac{1}{4}\alpha\kappa)}\right)(1 - r^2/a^2)\left\{1 + \frac{kr}{a}\cos\theta + \frac{\alpha k}{6}\left[\frac{3}{4}\left(1 + \frac{r^2}{a^2}\right) + \frac{r^2}{a^2}\cos 2\theta\right]\right\}, \tag{6}$$

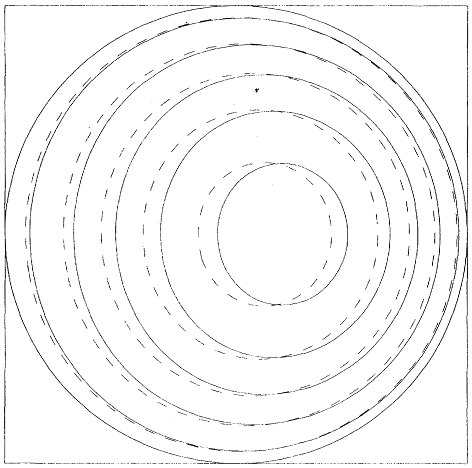

FIG. 1. Flux (solid lines) and constant-$\bar{p}$ surfaces (dashed lines) for equilibrium with $k = 0.5$ and $\alpha = -0.2$. The major axis lies to the left.

where $\bar{q} = 2\pi a^2 B_0/R_0 I_\varphi$ is a typical safety factor, I_φ being the toroidal current, and B_0 characterizes the magnetic field. Because of the ordering $p_\parallel \equiv p_\parallel^{(1)}(\psi) + O(\delta^2)$, the variation of $\bar{p}$ on the flux surfaces results from $p_\perp$ only. For given values of the free parameters k and α, the poloidal beta, defined by

$$\beta_p = 8\pi I_\varphi^{-2} \int \bar{p} r\, dr\, d\theta = \delta^{-1} k(1 + \tfrac{1}{3}\alpha k)(1 + \tfrac{1}{4}\alpha k)^{-2},$$

is fixed, and Eq. (6) determines an entire class of equilibria whose members are distinguished by a further parameter $\xi \geq |\alpha|/(1 - |\alpha|)$ such that $p_\perp/p_\parallel = \xi + \alpha(1+\xi)(r/a)\cos\theta$. Flux and pressure surfaces for a typical case are shown in Fig. 1. As will be made clear later, α is related to the intensity and angle of injection, and is generally small, so that ξ is a direct measure of the pressure anisotropy. Typically, $\frac{1}{3} \lesssim \xi \lesssim 3$ since the hot-ion pressure can be comparable with that of the background plasma for which $\beta \sim \delta$. Parallel injection into a low-pressure background plasma with $\beta \sim \delta^2$ can be modeled also, by setting $\xi = \alpha = 0$, in which case $p_\parallel \sim \delta$, $p_\perp \sim \delta^2$ and the equilibrium coincides to leading order with that of Cordey and Haas.[15] Irrespective of the value of ξ, it is clear from Eqs. (5) and (6) that equilibrium and stability properties depend only on the injection parameter α and on k, which determines β_p.

We have taken the large-aspect-ratio form of Eq. (5), and by varying $\partial\bar{p}/\partial\psi$ to satisfy the boundary condition on F as $y \to \infty$ have calculated numerically the pressure gradients at marginal stability for surfaces in the above equilibrium. Except

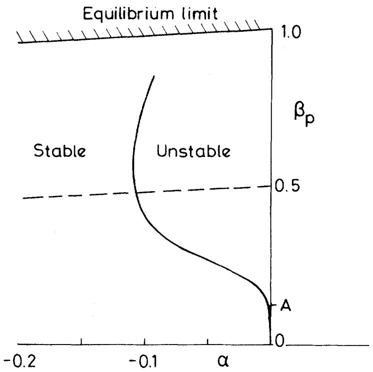

FIG. 2. Variation of the marginally stable poloidal β, measured in units of δ^{-1}, is plotted vs α. For β_p above the dashed line, the toroidal current reverses on the inside.

for the immediate vicinity of the magnetic axis, the localized interchange criterion is always satisfied when $\alpha \leq 0$. On a surface of given shape and magnetic field, with a prescribed amount of shear, there are, in general, two marginally stable pressure gradients which bound a range of unstable values. When $\alpha = 0$, the equilibrium value of $\partial\bar{p}/\partial\psi$ is found always to lie in the unstable range, but rather close to the higher marginal point. As α falls below zero, the unstable range narrows. Plotting β_p versus α, our results are presented as a marginal stability line in Fig. 2. We also indicate the equilibrium limit and, for completeness, the current-reversal limit. We observe that for very modest values of α (-0.1), our model is ballooning-stable right up to the equilibrium limit; as β_p approaches this limit, the value of α corresponding to marginal stability begins to decrease. The effect is thought to result from a stabilization associated with large values of the major-radius displacement. function Δ' close to the boundary, where $\Delta' \sim k$. The least stable surface is always near the boundary. In obtaining the marginal curve, we have excluded the magnetic axis and a small surrounding region (less than 1% of plasma volume). Any instability in the excluded region will thus be strongly localized round the axis and is therefore disregarded.

In the case of scalar pressure ($\alpha = 0$) as $k \to 0$ the shear at the boundary vanishes like k^2, and as a result our equilibrium is unstable even in the limit of small pressure. However, with a current profile producing finite shear at the boundary we expect stability up to a finite limit in β_p. When the additional shear is small this value is in the vicinity of point A in Fig. 2; the latter point is obtained by applying the result of Connor, Hastie, and Taylor[6] to our circular boundary.

As a consequence of large aspect ratio and Eq. (3), $\bar{p}$ can only exhibit lowest-order variation round a flux surface if C is $O(1)$. Calculations of $p_\perp$, $p_\parallel$, and C show that for a distribution function which models single-beam injection into the passing band,[16] such variation of $\bar{p}$ is possible only when the beam pressure is $O(\delta)$ and the angle of injection is large, so that hot ions are created with $v_\parallel/v = T\delta^{1/2}$, where $T(\psi)$ is $O(1)$ but sufficiently high to avoid trapping. (Ignoring the diamagnetic part of B_φ, this is ensured if $T > \sqrt{2}$ for injection onto a surface of circular cross section.) Furthermore, in the limit $\delta \ll 1/T^2 \ll 1$, the beam perpendicular pressure takes the form $P_{\perp b}(\psi)[1-(1/2T^2)(r/a)\cos\theta]$, there being no $O(\delta)$ hot-ion contribution to $p_\parallel(\psi)$. With linear forms for the isotropic background pressure $p_0(\psi)$ and for $P_{\perp b}(\psi)$, $\bar{p}$ now assumes the model form leading to Eq. (6) (provided T is constant) where ξ becomes $\xi = 1 + P_{\perp b}(\psi)/p_0(\psi)$, and $\alpha = -\frac{1}{2}(\xi-1)/(\xi+1)T^2$. Thus, ξ measures the strength of the injection source, and α depends on the angle of injection. Typically with $\xi \sim 3$ and $T \sim \sqrt{3}$, we have $\alpha \sim 0.1$, so the range of values covered in Fig. 2 is characteristic.

For the same class of current profiles, it is clear from Fig. 2 that by a modest inward weighting of pressure, a significant improvement in β can be obtained over the scalar-pressure value. Although the weighting modifies the shear, this effect is small at the values of α (-0.1) necessary to ensure stability up to the equilibrium limit. Naturally, we expect this class of profiles to be kink unstable; stabilization of this mode requires shaping of the current profile. We conjecture that the effect to which attention has been drawn in this Letter may also give rise to improved ballooning stability for equilibria possessing more realistic current profiles.

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[1]A. M. M. Todd, M. S. Chance, J. M. Greene, R. C. Grimm, J. L. Johnson, and J. Manickam, Phys. Rev. Lett. 38, 826 (1977).

[2]G. Bateman and Y.-K. M. Peng, Phys. Rev. Lett. 38, 829 (1977).

[3]A. Sykes, J. A. Wesson, and S. J. Cox, Phys. Rev. Lett. 39, 751 (1977).

[4]B. Coppi, Phys. Rev. Lett. 39, 938 (1977).

[5]D. Dobrott, D. B. Nelson, J. M. Greene, A. H. Glasser, M. S. Chance, and E. A. Frieman, Phys. Rev. Lett. 39, 943 (1977).

[6]J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. 40, 396 (1978).

[7]J. F. Clarke, ORNL Report No. ORNL/TM 5429, 1976 (unpublished).

[8]R. M. Kulsrud and D. L. Jassby, Nature (London) 259, 541–544 (1976).

[9]M. D. Kruskal and C. R. Oberman, Phys. Fluids 1, 275 (1958).

[10]J. Andreoletti, C. R. Acad. Sci. 256, 1251 (1963).

[11]P. H. Rutherford and L. Chen, unpublished.

[12]J. W. Connor and R. J. Hastie, Phys. Fluids 19, 1727 (1976).

[13]C. Mercier and M. Cotsaftis, Nucl. Fusion 1, 121 (1961).

[14]J. B. Taylor and R. J. Hastie, Phys. Fluids 8, 323 (1965).

[15]J. G. Cordey and F. A. Haas, Nucl. Fusion 16, 605 (1976).

[16]J. G. Cordey and M. H. Houghton, Nucl. Fusion 13, 215 (1973).