Resonant Effects on the Low-Frequency Vlasov Stability of Axisymmetric Field-Reversed Configurations

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We investigate the effect of particle resonances on low-frequency magnetohydrodynamic modes in field-reversed geometries, e.g., an ion ring. It is shown that, for sufficiently high field reversal, modes which are hydromagnetically stable can be driven unstable by ion resonances. The stabilizing effect of a toroidal magnetic field is discussed.

A renewed interest has developed in field-reversed or Astron-type magnetic confinement systems produced by the injection of beams of protons[1] or neutral hydrogen atoms into a magnetic mirror field.[2] If a sufficient current of injected ions can be trapped, a region of closed magnetic field lines will be created within a larger region of open field lines. Provided such a configuration has favorable stability properties, it could lead to a promising fusion reactor. The low-frequency ($\omega \ll \Omega_i$, the ion gyrofrequency) stability of a specific configuration, viz. an ion ring, against kink modes has been studied by Lovelace.[3] A more general approach has been developed by Sudan and Rosenbluth[4,5] to arrive at an energy principle in which the energetic beam ions are treated by the Vlasov formalism whereas the background plasma is described by the two-fluid equations.

In this paper we investigate the effects of beam particle resonances on the stability of low-frequency modes in a field-reversed system. Although such resonances are implicitly contained in the analysis of Sudan and Rosenbluth,[4,5] in the two cases studied in detail, the long P layer or θ pinch and the large-aspect-ratio ion ring, resonant instabilities have exponentially small growth rate for low beam temperature. However, for appreciable beam temperature, which is necessary for field reversal on axis for rings of finite axial extent, the energy absorbed by the resonant ions in the beam frame can be substantial, and negative energy modes in the beam frame can be driven to instability with growth rate approaching that of a hydromagnetic instability.

Assume that the energetic ions can be described by a rigid-rotor equilibrium $f_0(H - \Omega P_\theta)$, where H and P_θ are the particle energy and canonical momentum, respectively. The mean rotation frequency Ω is typically of the order of magnitude of the gyrofrequency in the external magnetic field, Ω_i, for large gyroradius particles. For an ion ring most beam ions are of this type. We give a brief derivation of the energy principle of Sudan and Rosenbluth[4,5] in a modified form. For a plasma displacement $\xi \sim \exp(il\theta - i\omega t)$, with $\omega \ll \Omega_i$, the linearized equation of motion of the background plasma is

$$-n_i m_i \omega^2 \vec{\xi} = -e_b n_b \delta\vec{E} + \delta\vec{j}_p \times \vec{B}/c\,, \tag{1}$$

where $\delta\vec{j}_p$ is the perturbed plasma current. The subscripts i, e, and b refer to background plasma ions, background electrons, and beam, respectively. We have assumed that the background plasma is cold and that there is no magnetic field in the toroidal, i.e., $\hat{\theta}$, direction. Therefore, no equilibrium plasma currents appear in (1), although the generalization to include these currents and B_θ is straightforward. [In our cylindrical coordinate system (r, θ, z), the beam current is in the $\hat{\theta}$ direction if $\Omega > 0$ and the external magnetic field is in the $-\hat{z}$ direction.] We have also used the charge neutrality condition $e_b n_b + e_i n_i + e_e n_e = 0$. Now from Ohm's law $\delta\vec{E} - i\omega\vec{\xi} \times \vec{B}/c = 0$, and Faraday's law, we obtain $\delta\vec{B} = \nabla \times (\vec{\xi} \times \vec{B})$. Employing this relation and $\nabla \times \delta\vec{B} = (4\pi/c)(\delta\vec{j}_p + \delta\vec{j}_b)$, where $\delta\vec{j}_p$ and $\delta\vec{j}_b$ are the perturbed plasma and bean currents, respectively, multiplying (1) by $\vec{\xi}^*$ and integrating over the total volume, we obtain

$$-\omega^2 M + \delta W = \omega L + \int d^3x\, \delta\vec{j}_b \cdot \delta\vec{A}^*/2c, \tag{2}$$

where

$$\delta\vec{A} = \xi \times \vec{B}\,,$$

$$M = \tfrac{1}{2}\int d^3x\, n_i m_i\, |\vec{\xi}|^2\,,$$

$$\delta W_m = \tfrac{1}{2}\int d^3x\, |\delta\vec{B}|^2/4\pi\,,$$

and

$$L = -(ie_b/2c)\int d^3x\, n_b \vec{B} \cdot \vec{\xi}^* \times \vec{\xi}\,.$$

From the Vlasov equation we obtain the perturbed distribution function[4,5] $\delta f = (e_b f_0'/c) \times (-\Omega r \delta A_\theta + g)$, where $f_0' \equiv \partial f_0/\partial H$, $dg/dt = -i(\omega - l\Omega)\vec{v} \cdot \delta\vec{A}$, and d/dt is the derivative along unperturbed trajectories. Substituting $\delta\vec{j}_b = e_b \int d^3v\, \vec{v}$

δf, we find

$$(1/2c)\int d^3x\, \delta\vec{j}_b \cdot \delta\vec{A}^* = -(e_b{}^2/2c^2)[\Omega^2 \int d^3x\, d^3v\, r^2 |\delta A_\theta|^2 f_0{}' + i(\omega^* - l\Omega)^{-1}\int d^3x\, d^3v f_0{}' g\, dg^*/dt]. \tag{3}$$

The beam particles in general have quite complex trajectories in a fully field-reversed system in which the radial and axial widths are of the same order of magnitude and the beam temperature is more than a few percent of the directed beam energy. In fact, it has been shown that in this case no invariant of motion besides P_θ and H exists.[6] In customary usage, the orbits are ergodic or stochastic on each (P_θ, H) surface in phase space. In this case, direct integration over unperturbed trajectories to obtain the last term in (3) is difficult, if not impossible. However, invoking the stochasticity of these orbits we proceed by expressing

$$I \equiv \int d^3x\, d^3v f_0{}' g\, dg^*/dt = m_b{}^{-2}\int dP_\theta\, dH\, V(P_\theta, H) f_0{}' \langle g\, dg^*/dt\rangle, \tag{4}$$

where $V(P_\theta, H)$ is the four-dimensional volume in phase space accessible to a particle with given P_θ and H, and angular brackets denote an average over this volume. We write $\theta(t) = \overline{\Omega}(P_\theta, H)t + \tilde{\theta}(t)$, where $\overline{\Omega}$ is the time average of $\dot{\theta} = (P_\theta - erA_\theta/c)/2m_b r^2$. We conclude that $\overline{\Omega}$ is a function of P_θ and H alone by invoking the ergodicity property and replacing the time average of $\dot{\theta}$ by its phase average over V. Introducing the notation $\vec{v}\cdot\delta\vec{A} = K(t)\exp(-i\alpha t)$, where $\alpha = \omega - l\Omega$, we obtain

$$\begin{aligned}\langle g\, dg^*/dt\rangle &= |\omega - l\Omega|^2 \langle K^*(t)\exp(i\alpha^* t)\int_{-\infty}^{t} dt' K(t')\exp(-i\alpha t')\rangle \\ &= |\omega - l\Omega|^2 \exp(i\alpha^* t)\int_{-\infty}^{t} dt' \exp(-i\alpha t')\langle K^*(t)K(t')\rangle.\end{aligned} \tag{5}$$

By the ergodic property again, the phase average in (5) may be replaced by a time average. Thus it is clear that this term, called the phase-space autocorrelation, is a function of $t - t'$ alone. However, the ergodic property is not a necessary condition. In fact, in the limiting cases of low beam temperature and special geometry when the equations of motion are nearly linear, $\overline{\Omega}$ is a function of P_θ alone and the autocorrelation is a function of $t - t'$. We may therefore express it as the Fourier transform of a power spectrum $P(\nu)$ and it is clear that the very complex phase information in the individual trajectories is irrelevant. We find

$$\langle g\, dg^*/dt\rangle = |\omega - l\Omega|^2 \exp(2\gamma t)\int_{-\infty}^{\infty}(d\nu/2\pi i) P(\nu)(\nu - \alpha)^{-1}, \tag{6}$$

where $\gamma = \mathrm{Im}\,\omega$.

Substituting (6), (4), and (3) in (2), we find

$$\omega^2 M + \omega L - \delta W + iR = 0, \tag{7a}$$

where

$$\delta W = \delta W_m + (e_b{}^2\Omega^2/2c^2)\int d^3x\, r^2 |\delta A_\theta|^2 \int d^3v f_0{}' + (e_b{}^2/4\pi m_b{}^2c^2)(\omega - l\overline{\Omega})\int dP_\theta\, dH\, V(P_\theta, H) f_0{}' \times \mathrm{Re}\int d\nu\, P(\nu)/(\nu - \alpha) \tag{7b}$$

and

$$R = -(e_b{}^2/4\pi m_b{}^2c^2)(\omega - l\overline{\Omega})\int dP_\theta\, dH\, V(P_\theta, H) f_0{}' \,\mathrm{Im}\int d\nu\, P(\nu)/(\nu - \alpha). \tag{7c}$$

[The common factor $\exp(2\gamma t)$ has been canceled.] For $\omega \ll \Omega_i$, we can neglect L, because it is of order n_b/n_i, and the frequency dependence of δW and R, because $\omega \ll l\Omega$. The power spectrum $P(\nu)$ has been computed for several cases and is found, as expected, to be peaked at the radial and axial betatron frequencies.[6] The betatron frequency is the frequency of the single–beam-particle motion in the self-magnetic field of the beam. In the low-frequency limit δW and R are real quantities and R represents the energy absorbed by the resonant particles, i.e., the particles for which the Doppler-shifted frequency $\omega - l\Omega \simeq -l\Omega$ coincides with their betatron frequencies.

Given a normal-mode displacement $\vec{\xi}(\vec{x})$ the coefficients in (7) can be computed so that the mode frequency is given by

$$\omega = \pm[(\delta W - iR)/M]^{1/2}. \tag{8}$$

The two values of ω represent a splitting of the degeneracy between the two modes, one with phase velocity in the direction of the beam current, and the other with phase velocity opposite to the beam current. It is easily seen that, for $\delta W>0$ and $R\neq 0$, the former mode is unstable whereas the latter mode is damped. If $\delta W<0$, the mode is a hydromagnetic (as opposed to resonant) instability and the resonant term R to lowest order affects only the real part of the frequency. Note that ω is in general complex, since the system is not self-adjoint. Also note that it is not necessary (or even possible) to obtain the stability criterion by minimizing the energy with respect to $\vec{\xi}$. In fact, instability occurs as long as $R\neq 0$, i.e., there is no absolute finite–Larmor-radius stabilization.[7,8]

As an example, consider a rigid kink mode $\vec{\xi}=\xi_0\hat{r}\exp(il\theta-i\omega t)$ in a large-aspect-ratio ion-ring system $r_0/a\gg 1$. (Here, r_0 is the major radius and a is the minor radius.) Assume that $\vec{B}=B_0\rho a^{-1}\hat{\varphi}$, where $\rho^2=(r-r_0)^2+z^2$ and $\varphi=\tan^{-1}[(r-r_0)/z]$. The orbits in this system are purely harmonic for low enough energy, and the autocorrelation function behaves as $\cos\omega_\beta(t-t')$ with $\omega_\beta^2=e_bB_0r_0\Omega/m_bca$. For $l\neq 0$ we find

$$P(\nu)\simeq r_0^2\Omega^2\xi_0^2B_0^2\langle\rho^2\rangle[\delta(\nu-\omega_\beta)+\delta(\nu+\omega_\beta)]/2a^2, \tag{9}$$

$$\operatorname{Im}\int d\nu\,P(\nu)/(\nu-\alpha)\simeq\pi P(l\overline{\Omega}),\quad \operatorname{Re}\int d\nu\,P(\nu)/(\nu-\alpha)\simeq-2r_0^2\Omega^2l\overline{\Omega}\xi_0^2B_0^2\langle\rho^2\rangle/2a^2(\omega_\beta^2-l^2\overline{\Omega}^2). \tag{10}$$

As shown in Refs. 3 and 5, the kink is hydromagnetically unstable, i.e., $\delta W<0$, if $\omega_\beta^2>l^2\Omega^2$. In both papers the authors found that the dominant part of δW behaved as $(l^2\Omega^2-\omega_\beta^2)^{-1}$. We find, from (10) and (7b), that this is true, except in a narrow region determined by the thermal spread of the beam. Thus we recover the same stability criterion as long as resonances are ignored. That is, modes with $|l|<l_0\equiv(r_0/a)(\zeta/\pi)^{1/2}$, where ζ is the field-reversal factor $\Delta B/B$, evaluated at $r=z=0$, are unstable, whereas those with higher l are stable.

When resonances are included, these conclusions are somewhat modified. Using (9) and assuming an exponential rigid-rotor $f_0\sim\exp[-(H-\Omega P_\theta)/T]$, we find

$$R\simeq\tfrac{1}{4}\pi^{3/2}(|\xi_0|^2)B_0^2r_0(E/T)^{1/2}\exp[-(E/T)(1-l_0^2/l^2)^2], \tag{11}$$

where $E=\frac{1}{2}mR^2\Omega^2$ is the directed beam energy and for the "bicycle-tire" geometry $\zeta/2\pi\simeq T/E$. Thus it is clear that modes with $|l|\sim l_0$ are most affected by resonances. Since $\delta W_m\sim\xi_0^2B_0^2r_0$, we see that modes with $|l|>l_0$ have a growth rate which is exponentially small in comparison with hydromagnetic modes if $(T/E)r_0/a\simeq(\zeta/2\pi)r_0/a$ is less than unity. On the other hand, if this parameter is of order unity, modes with $l^2\lesssim l_0^2(1+T/E)$ have growth rate comparable to that of a hydromagnetic instability.

Finally, we investigate the effect of a toroidal magnetic field on these resonant instabilities. For simplicity we consider a straight beam with constant axial current density $j_0\hat{s}$ and constant axial field $B_T\hat{s}$. For motion perpendicular to the beam current,

$$\ddot{x}+\omega_\beta^2x-\omega_T\dot{y}=0,\quad \ddot{y}+\omega_\beta^2y+\omega_T\dot{x}=0, \tag{12}$$

where $\vec{B}=B_0a^{-1}(x\hat{y}-y\hat{x})+B_T\hat{s}$, $B_0=2\pi j_0a/c$, $\omega_\beta^2=eB_0V_s/mca$, and $\omega_T=eB_T/mc$. We assume the axial velocity V_s is constant. The two frequencies of motion are given by $\omega_\pm=\frac{1}{2}\omega_T\pm[(\frac{1}{2}\omega_T)^2+\omega_\beta^2]^{1/2}$. As $\omega_T\to 0$ we recover $\omega_\pm=\pm\omega_\beta$ and as $\omega_T/\omega_\beta\to\infty$ we find $\omega_+\to\omega_T$, $\omega_-\to-\omega_\beta^2/\omega_T$. Therefore, for a fairly large toroidal field $B_T/B_0\gtrsim 1$, the two frequencies of motion split and the resonance condition no longer holds. More specifically, the $\omega-l\Omega=-\omega_+$ resonance occurs at increasing values of l as B_T increases. However, the growth rate of the mode nearest resonance scales as $1/lB_T\sim 1/B_T^2$ for large B_T. The $\omega-l\Omega=\omega_-$ resonance occurs for decreasing values of l as B_T increases and disappears altogether when $B_T/B_0>R/a$.

The existence of this instability for $B_T=0$ and its stabilization for large B_T may explain the experimental results of Davis, Meger, and Fleischmann on RECE-Christa,[9] where a toroidal field $B_T\gtrsim B_0$ is required to trap and stabilize the electron ring.

The above treatment is general enough to include field-reversed configurations with arbitrary ion gyroradius. On the completion of this work we learned that Seyler and Freidberg[8] have used a similar formalism to examine resonant effects in the special case of an infinitely long θ pinch in the small-gyroradius approximation.

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Interfacial Profile in Three Dimensions

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Results of a first-principles renormalization-group calculation of the three-dimensional critical interface in a gravitational field are presented. No artificial cutoffs to prevent divergences are required. A detailed comparison with reflectivity measurements of Wu and Webb is included.

The nature of the density (or composition) profile of the diffuse interface separating two coexisting phases has been a topic of considerable interest for some time.[1] Recent theoretical developments have brought the question into sharp focus[2,3]; furthermore, one finds aspects of the problem in common with those found in other areas of current theoretical activity. The crucial feature any theory must deal with is the *continuous symmetry* which corresponds to translation of the nominal interface center. The associated long-wavelength fluctuations, which correspond to local translations, have sufficient density[4] in $d=3$ to require a cutoff for a proper definition of the interface width (even far below T_c).

A combination of mean-field and more refined phenomenological scaling arguments[5] have led to the expectation that the *critical* profile is universal. Universality in this case is taken to mean that for, say, a single-component fluid one has density given by

$$\rho(z)=\frac{\rho_l+\rho_v}{2}-\frac{\rho_l-\rho_v}{2}m(z/\xi),$$

where $\rho_{l(v)}$ are the bulk liquid (vapor) densities and ξ is the bulk correlation length. The same function $m(z)$ is to describe the interface for all fluids, binary liquids, anisotropic magnets, etc. The expectation of universality has been reinforced by recent calculations[2,3] of the critical profile in $4-\epsilon$ dimensions. Furthermore, experiments[6,7] have not revealed any systematic nonuniversalities. However, the presence of the above class of long-wavelength fluctuations considerably influences the nature of the criticality and, in fact, causes the breakdown of the ϵ expansion in bulk dimensionality $d=3$ ($\epsilon=1$). Hence one must view with caution phenomenological predictions of universality in $d=3$.

An alternative semiphenomenological approach to the critical interface is contained in the work of Buff, Lovett, and Stillinger,[8] who consider *only* the contribution of the above-mentioned fluctuations while neglecting the role of ordinary critical fluctuations. Their work in fact implies a nonuniversal profile having a width that diverges in the limit of vanishing gravitational field.

In this Letter we report the results of a calculation of the critical interface in $d=3$ in which we treat all fluctuations within one formalism. At the end it is observed that the effects of the "capillary wavelike" fluctuations separate from the effects of other "ordinary" critical fluctua-