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Magnetostatic Mode and Cross-Field Electron Transport

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A zero-frequency magnetostatic mode is shown to exist in a magnetized plasma. This mode resembles the two-dimensional electrostatic convective-cell mode in many ways. Electron cross-field test-particle diffusion due to thermally excited magnetostatic modes exhibits the Bohm-like T/B behavior. This mode would enhance the electron heat and momentum transport and could permit rapid spreading of plasma current.

Since the discovery of the zero-frequency electrostatic convective-cell (vortex) mode[1] in two-dimensional plasma, remarkable advances have been made on the theory of plasma transport by collective processes.[2-5] It has been demonstrated, both computationally and analytically, that even for a plasma in thermal equilibrium the long-lived, large-scale, low-frequency fluctuating electric field can cause convection of the plasma by $\vec{E}\times\vec{B}$ drifts across the externally applied confining magnetic field and thus cause cross-field plasma transport well above the classical collisional value. Evidence of plasma transport by convective cells has also been observed experimentally.[6,7] In this paper, we show that besides the electrostatic convective cell, a zero-frequency magnetostatic mode can also exist in a magnetized plasma. In the plane perpendicular to the externally applied static magnetic field, the particle motion (resulting from streaming along the perturbed magnetic field line) of this mode resembles that of the $\vec{E}\times\vec{B}$ motion of the convective cell. Similar to the convective cell, the long-wavelength magnetostatic mode can persist for a long time. The cross-field test-particle diffusion from thermally excited magnetostatic modes exhibits the Bohm-like T/B scaling.

For a uniform plasma immersed in a constant magnetic field $\vec{B}_0 = B_0\hat{z}$ there are two types of modes which can propagate across the ambient magnetic field ($k_z=0$): (1) the extraordinary mode[8] with dispersion relation $N^2=RL/S$ and (2) the ordinary mode[8] with dispersion relation $N^2=P$ (the notations are the same as those defined in Ref. 8). Electrostatic waves are a subset of the extraordinary mode in the limit that N^2 approaches infinity ($S\to 0$), and the convective cell is one of them. The second type of mode, the ordinary mode, is a purely transverse electromagnetic wave: The wave vector, the wave electric field (which is parallel to the external magnetic field), and the wave magnetic field form a right-handed orthogonal set, while the charge-density variation is zero and only the $\hat{z}$ component of the vector potential $\vec{A}$ is involved. As we will show below, a zero-frequency mode similar to the electrostatic convective cell exists in this type of transverse electromagnetic wave. The technique developed in Ref. 3 can be used here; however, because of the complexity of the mathematics involved, it seems more appropriate to use a heuristic approach instead. The equations to be used in the analysis are the wave equation for the perturbation A_z,

$$\nabla^2 A_z - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\frac{4\pi}{c} j_z, \tag{1}$$

and the electron momentum equation in the z direction,

$$\partial v_z/\partial t + \vec{v}\cdot\nabla v_z = -(q/m)E_z + \mu\nabla^2 v_z - \nu v_z, \tag{2}$$

where c is the speed of light, ν is the electron-ion collision frequency, and μ is the collective shear viscosity[9] which will be calculated later when we consider the particle motion in the plane perpendicular to $\hat{z}$. Because of their considerably smaller charge-to-mass ratio, the ion v_{zi} response can be neglected; therefore

$$j_z = ne(v_{zi} - v_z) \simeq -nev_z. \tag{3}$$

The perturbed field quantities are related to A_z through the following relations:

$$\vec{B}_1 = \nabla \times (A_z \hat{z}) = \nabla A_z \times \hat{z}, \tag{4}$$

and

$$\vec{E}_1 = -c^{-1}(\partial A_z / \partial t)\hat{z}. \tag{5}$$

Linearizing Eq. (2), and assuming a phasor $\exp[i(\vec{k}\cdot\vec{r} - \omega t)]$ for the perturbed quantities, we obtain the following dispersion relation:

$$N^2 = \epsilon(\vec{k}, \omega), \tag{6}$$

with

$$\epsilon(\vec{k}, \omega) = 1 - \omega_p^2/\omega[\omega + i(\nu + \mu k^2)], \tag{7}$$

where $N^2 = k^2c^2/\omega^2$ is the square of the refraction index and $\omega_p^2 = 4\pi ne^2/m$ is the plasma frequency. This dispersion relation allows both a well-known high-frequency ordinary mode,[8]

$$\omega = \pm(\omega_p^2 + k^2c^2)^{1/2} - \frac{1}{2}\frac{i(\nu + \mu k^2)\omega_p^2}{\omega_p^2 + k^2c^2}, \tag{8}$$

and an extremely low frequency, purely damped mode,

$$\omega = -\frac{i(\nu + \mu k^2)}{1 + \omega_p^2/k^2c^2}. \tag{9}$$

(The low-frequency, purely damped mode in an *unmagnetized* plasma has been discussed by Ginzburg and Ruhadze.[10]) For a plasma in thermal equilibrium, the power spectra for fluctuating magnetic field and electric field can be calculated from the fluctuation-dissipation theorem[11] and are given by

$$\left\langle \frac{B^2(\vec{k}, \omega)}{8\pi} \right\rangle = \frac{T}{2\pi}\frac{1}{\omega}\,\mathrm{Im}\left(\frac{k^2c^2}{k^2c^2 - \omega^2\epsilon(\vec{k}, \omega)}\right), \tag{10}$$

and

$$\left\langle \frac{E^2(\vec{k}, \omega)}{8\pi} \right\rangle = \frac{T}{2\pi}\frac{1}{\omega}\,\mathrm{Im}\left(\frac{\omega^2}{k^2c^2 - \omega^2\epsilon(\vec{k}, \omega)}\right). \tag{11}$$

where T is the temperature in energy units, both ω and k are treated as real, and Im means taking the imaginary part. Integrating expressions (10) and (11) with respect to frequency over the high-frequency ordinary mode in Eq. (8), we obtain the energy in magnetic field and electric field per $\vec{k}$:

$$\langle B^2/8\pi \rangle_{k,h} = \tfrac{1}{2}Tk^2c^2/(\omega_p^2 + k^2c^2), \tag{12}$$

$$\langle E^2/8\pi \rangle_{k,h} = \tfrac{1}{2}T, \tag{13}$$

where the subscript h denotes the higher-frequency mode. The kinetic energy of the electron v_z motion in the wave is

$$\langle \tfrac{1}{2}nmv_z^2 \rangle_{k,h} = \tfrac{1}{2}T\omega_p^2/(\omega_p^2 + k^2c^2). \tag{14}$$

At this point, it is worth mentioning that the energy in the mode is independent of the damping mechanism for a plasma in thermal equilibrium. The damping will affect the width of the spectrum but not the total energy in it. This is true for all modes in a thermal plasma including the low-frequency mode of Eq. (9). Adding up Eqs. (12)–(14), and dividing by 2 (two polarizations, $\omega > 0$ and $\omega < 0$), we have $T/2$ of wave energy in each mode. Integrating the field energy over the low-frequency part (denoted by subscript l) of the spectrum yields

$$\langle B^2/8\pi \rangle_{k,l} = \tfrac{1}{2}T\omega_p^2/(\omega_p^2 + k^2c^2), \tag{15}$$

$$\langle E^2/8\pi \rangle_{k,l} = 0, \tag{16}$$

and the wave kinetic energy in the perturbed electron v_z motion is

$$\langle \tfrac{1}{2}nmv_z^2 \rangle_{k,l} = \tfrac{1}{2}Tk^2c^2/(\omega_p^2 + k^2c^2). \tag{17}$$

We see that the wave energy in the low-frequency mode is also $T/2$. In addition, summing up the contribution from the high-frequency mode and low-frequency mode, we have $T/2$ energy in both the electric field and magnetic field per $\vec{k}$. This result agrees with the prediction of equilibrium statistical mechanics.[12]

The zero-frequency electromagnetic mode has negligible electric fields, and so we may call it a magnetostatic mode. Only a fraction, $(1 + \omega_p^2/k^2c^2)^{-1}$, of the total wave energy of the magnetostatic mode is in the electron v_z motion, and collisions and viscosity dissipate only this part of the energy. It is understandable why the factor $(1 + \omega_p^2/k^2c^2)^{-1}$ occurs in the damping expression in Eq. (9). Since the particle energy is small and viscous damping is weak for the long-wavelength mode, the large-scale static magnetic field perturbation can have an extremely long lifetime. Therefore, the rapid particle motion along the perturbed field should enhance the plasma transport across the externally applied magnetic field.

We now estimate this effect by considering the single test-particle motion due to this magnetostatic mode in the plane perpendicular to the external magnetic field $B_0\hat{z}$. In the absence of perturbation, on a time scale much longer than the particle gyromotion, the particle effectively moves freely along the field line with some velocity v_0 but with no mobility across the magnetic field. In the field of the magnetostatic mode, the linearized equation of motion perpendicular to $\hat{z}$ is

$$\partial\vec{v}_1/\partial t = \Omega(\vec{v}_1\times\hat{z} + v_0\hat{z}\times\vec{B}_1/B), \tag{18}$$

where Ω is the cyclotron frequency. Since $\partial/\partial t \ll |\Omega|$ for the magnetostatic mode, Eq. (18) yields

$$\vec{v}_1 = v_0\vec{B}_1/B_0 = (v_0/B_0)(\nabla A_z\times\hat{z}). \tag{19}$$

This is simply a mathematical statement of the fact that the particle moves along the perturbed field line and thus acquires a velocity perpendicular to the external magnetic field $B_0\hat{z}$. Note the resemblance of this type of motion to the $-c(\nabla\varphi\times\hat{z})/B$ motion for particles in the electrostatic convective cell.

The test-particle diffusion rate can be calculated by evaluating the time integral of the velocity correlation function along the particle trajectory[13]:

$$\begin{aligned} D &= \int_0^\infty\langle\vec{v}_1(t)\cdot\vec{v}_1(t+\tau)\rangle\,d\tau, \\ &= (v_0^2/B_0^2)\int_0^\infty\langle\vec{B}_1(t)\cdot\vec{B}_1(t+\tau)\rangle\,d\tau. \end{aligned} \tag{20}$$

By adoption of the commonly used diffusing orbital theory,[14,15] the magnetic field correlation can be approximated by

$$\langle\vec{B}_1(t)\cdot\vec{B}_1(t+\tau)\rangle = (2\pi)^{-3}\int\langle B^2(\vec{k},\omega)\rangle\exp(-Dk^2\tau + i\omega\tau)\,d\omega\,d^3k = (2\pi)^{-3}\int\langle B^2\rangle_{\vec{k},l}\exp(-Dk^2\tau)\,d^3k. \tag{21}$$

Because the contribution from the high-frequency ordinary mode is negligible, only the low-frequency spectrum is kept in Eq. (21). Substituting the low-frequency energy in the thermal spectrum [as shown in Eq. (15)] for $\langle B^2\rangle_{\vec{k},l}$ we have[1,2,5]

$$D^2 = \frac{v_0^2}{B_0^2}\frac{2T}{L_\parallel}\ln\left(\frac{L\omega_p}{2\pi c}\right), \tag{22}$$

where $L_\parallel$ and L are the linear dimensions parallel and perpendicular to $\hat{z}$, respectively. If we average this expression over the Maxwellian equilibrium distribution function, v_0^2 becomes T/m, and we then have

$$D_M = \frac{T}{B_0}\left[\frac{2}{mL_\parallel}\ln\left(\frac{L\omega_p}{2\pi c}\right)\right]^{1/2}. \tag{23}$$

This test-particle diffusion coefficient has the Bohm-like T/B scaling.

Note again the similarity between the diffusion coefficient derived here and that due to the convective cell in the guiding-center limit ($\omega_{pi}\ll\Omega_i$),

$$D_c = \frac{cT^{1/2}}{B_0}\left[\frac{2}{L_\parallel}\ln\left(\frac{L}{2\pi\lambda_D}\right)\right]^{1/2}, \tag{24}$$

where λ_D is the Debye length. In fact, the magnetostatic mode and guiding-center convective cell are analogous to each other. The correspondence of the variables is current $\to$ charge and magnetic potential $A_z \to$ electrical potential φ.

At this point we would like to point out also some differences between the two modes. First, for a thermal-equilibrium plasma, the energy in the magnetostatic mode is independent of the strength of the external magnetic field B_0. On the other hand, the energy in the convective cell is proportional to $(1+\omega_{pe}^2/\Omega_e^2+\omega_{pi}^2/\Omega_i^2)^{-1}$ which is a function of B_0. In general, the diffusion coefficient in Eq. (24) has to be multiplied by a factor $(1+\omega_{pe}^2/\Omega_e^2+\omega_{pi}^2/\Omega_i^2)^{-1/2}$. Second, in a collisionless plasma ($\nu=0$), the lifetime of the long-wavelength magnetostatic mode scales like k^{-4}, whereas that of the convective cell goes like k^{-2}. Third, the diffusion rate for the convective cell is independent of the velocity of the particles. In the magnetostatic mode, however, particles with high parallel velocity diffuse faster than those with low parallel speed. In particular, electrons tend to diffuse much faster than ions. Since the true particle diffusion has to be ambipolar in order to maintain charge neutrality, the magnetostatic mode, like the lower hybrid mode, will only give rise to enhanced electron shear viscosity μ and electron heat conductivity κ but not particle diffusion. Or, equivalently, the magnetostatic mode will cause only turbulent electron-electron collisions.[16] Therefore, we may hypothesize that

$$\mu = D_M, \tag{25}$$

$$\kappa = nD_M. \tag{26}$$

In general, the contribution to the test-particle diffusion from other portions of the spectrum (e.g. convective cell, lower hybrid wave, etc.) should be added to D_M. The contribution from the lower hybrid wave is difficult to obtain analytically and will not be discussed here. We shall only compare the contributions from the convective cell and the magnetostatic mode. The ratio of the two test-particle diffusions for a thermal equilibrium plasma is

$$\frac{D_M}{D_c}=\left(\frac{T}{mc^2}\right)^{1/2}\left(1+\frac{\omega_{pe}^2}{\Omega_e^2}+\frac{\omega_{pi}^2}{\Omega_i^2}\right)^{1/2}\left(1+\frac{1}{2}\frac{\ln(T/mc^2)}{\ln(L/2\pi\lambda_D)}\right)^{1/2}. \tag{27}$$

In a super strong magnetic field ($\Omega_i^2 \gg \omega_{pi}^2$), the contribution from convective cells is dominant over that from magnetostatic modes by a factor $(T/mc^2)^{1/2}$. In a modestly strong magnetic field ($|\Omega_e| \simeq \omega_{pe}$), as in most tokamaks, the contribution of magnetostatic modes would be larger than that of convective cells for $\beta > m_e/M_i$, where $\beta = 4\pi nT/B^2$. Comparing D_M with the classical collisional contribution $D_\nu = \nu_e \rho_e^2$, where ρ_e is the electron Larmor radius, we have

$$\frac{D_M}{D_\nu}=3\left|\frac{\Omega_e}{\omega_{pe}}\right|\left(\frac{T}{mc^2}\right)^{1/2}\left(\frac{4\pi n\lambda_D^4}{L_\parallel}\right)^{1/2}\frac{[\ln(L\omega_p/2\pi c)]^{1/2}}{\ln\Lambda}, \tag{28}$$

where $\ln\Lambda$ is the Coulomb logarithm. This ratio is the order of unity for a wide range of plasma parameters. For example, for $n = 3\times 10^{14}/\mathrm{cm}^3$, $B = 4$ T, $L = 100$ cm, $L_\parallel = 2000$ cm, and $T = 10$ keV, D_M/D_ν is 1.1. We have to keep in mind that this comparison is valid only for fluctuating fields at the thermal level. In a confined plasma, the amplitude of electromagnetic fluctuations can easily exceed the thermal level by several orders of magnitude even for a very quiescent plasma.[6] Therefore, it is possible that in fusion devices the collective electron cross-field transport would be much more pronounced than that discussed above and could result in enhanced electron heat conductivity and rapid spreading of current. Generalization to the case of current-carrying plasma is underway and the results will be reported in a forthcoming paper.

In conclusion, we have shown that a magnetostatic mode exists in a magnetized plasma. Electron cross-field test-particle diffusion due to thermally excited magnetostatic modes exhibits Bohm-like T/B behavior. This mode enhances the electron cross-field heat conductivity and viscosity.

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