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Example of Color Screening

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An example is given of an admissible configuration of SU(3) gauge fields that completely screens an intense source of color and that has a lower energy than the corresponding Coulomb configuration.

Quantum chromodynamics has become the leading theory of the strong interactions because of its successful interpretations of scattering data and of hadron spectra. The absence of quarks and gluons is thought to be due to its being an unbroken non-Abelian gauge theory with a running coupling constant that increases with distance. If this is the reason for confinement, then some sign of it should be visible on the classical level.

The natural starting point for a classical explanation of confinement is Gauss's law which constrains the response of a gauge field to a charge. Yet one of the allowed responses in both Abelian and non-Abelian theories is the Coulomb configuration. Because Coulomb forces diminish with the square of the distance, they do not lend themselves to an explanation of quark confinement, although they do not exclude one.

It has been shown, however, by Mandula[1] that the Coulomb configuration is classically unstable in an unbroken SU(3) gauge theory when the external color charge $g^2Z/4\pi$ exceeds $\frac{3}{2}$. It is therefore possible that the actual response of a gauge field to a strong charge may not be the Coulomb solution and that the forces between strong charges may have a different dependence on the distance between them.

The present paper presents some evidence in support of this possibility. An example is given of a configuration of SU(3) gauge fields that (i) satisfies Gauss's law, (ii) completely screens a strong color charge, and (iii) has a lower energy than the corresponding Coulomb solution when the charge $g^2Z/4\pi$ is greater than 5.6.

The field equations satisfied by the field strengths

$$F_a^{\mu\nu} = \partial^\mu A_a^{\ \nu} - \partial^\nu A_a^{\ \mu} + f_{abc} A_b^{\ \mu} A_c^{\ \nu} \tag{1}$$

of an SU(3) gauge theory in the presence of an external color current $j_a^{\ \mu}$ are

$$D_\nu^{\ a} F^{\mu\nu} = \partial_\nu F_a^{\ \mu\nu} + f_{abc} A_\nu^{\ b} F_c^{\ \mu\nu} = g^2 j_a^{\ \mu}. \tag{2}$$

The structure constants of SU(3) are represented by the antisymmetric form f_{abc} and g is a coupling constant. For $\mu = 1, 2, 3$ these equations are the equations of motion that govern the time evolution of initial configurations of gauge fields. But for $\mu = 0$, Eq. (2) is a restriction on what initial configurations are admissible. This constraint, known as Gauss's law, requires the divergence of the color-electric field $E_a^{\ i} = F_a^{\ 0i}$ to equal the total color density which is the sum of $g^2 j_a^{\ 0}$ and $f_{abc} A_b^{\ i} E_c^{\ i}$:

$$\nabla \cdot \vec{E}_a = g^2 j_a^{\ 0} + f_{abc} \vec{A}_b \cdot \vec{E}_c. \tag{3}$$

It is the last term in this equation that allows for the possibility of color screening.

The energy of the system is given by the Hamiltonian

$$H = 2g^{-2}\int d^3x(\vec{E}_a^{\,2} + \vec{B}_a^{\,2}), \tag{4}$$

where $B_a^{\ i} = \frac{1}{2}\epsilon_{ijk}F_a^{\ jk}$ is the color-magnetic field. The term $f_{abc}\vec{A}_b\cdot\vec{E}_c$ may be used to screen the charge $g^2 j_a^{\ 0}$ thus diminishing its electrostatic energy. Because the gauge fields $A_a^{\ \mu}$ are massless, the additional energy due to the creation of the screen is merely its electric and magnetic energy.

Suppose that the external color current $j_a^{\ \mu}$ is static and spherically symmetric and that only its third component is nonzero:

$$j_a^{\ \mu}(\vec{x}, t) = \delta_0^{\ \mu}\delta_{a,3}\rho(r). \tag{5}$$

Then the Coulomb solution to the field equations consists of a radial, static, color-electric field $\vec{E}_3(x,t) = E_c(r)\hat{r}$ that satisfies the differential equation

$$E_c' + 2E_c/r = g^2\rho, \tag{6}$$

where the prime means d/dr. All other fields are zero except for $A_3^{\ 0}$ of which $\vec{E}_3$ is the gradient. If ρ is of the form

$$\rho = ar^2 e^{-u}, \tag{7}$$

where $u = r/r_0$, then $E_c(r)$ is given by

$$E_c(r) = 24g^2 a r_0^{\ 5} r^{-2} \times[1 - e^{-u}(1 + u + u^2/2 + u^3/6 + u^4/24)]. \tag{8}$$

The energy of the Coulomb solution is

$$H_c = (2g^2)^{-1}\int d^3x\, E_c^{\ 2} = (837/4)\pi a^2 g^2 r_0^{\ 9}. \tag{9}$$

The external charge $Z = \int \rho\, d^3x$ has the value $Z = 96\pi a r_0^{\ 5}$. In terms of Z, the Coulomb energy H_c is

$$H_c = \frac{94}{1024}\frac{g^2Z^2}{4\pi r_0}. \tag{10}$$

A field configuration that satisfies Gauss's law and that has a lower energy than the Coulomb configuration for $Z > 5.61$ will now be exhibited. This will be done by exciting the fields $\vec{E}_1$ and $\vec{A}_2$ in such a way that the external charge $g^2 j_3^{\ 0}$ is canceled on every spherical shell. If all other fields except $\vec{E}_3$ are zero, then Gauss's law (3) reduces to the pair of equations

$$\nabla\cdot\vec{E}_1 = \vec{A}_2\cdot\vec{E}_3 \tag{11}$$

and

$$\nabla\cdot\vec{E}_3 = g^2\rho - \vec{A}_2\cdot\vec{E}_1. \tag{12}$$

If the vectors $\vec{E}_1$ and $\vec{A}_2$ are taken to be

$$\vec{E}_1(r,\theta,\varphi,0) = (3ar_0)^{1/2} g r \sin\theta\, e^{-u/2}\hat{\varphi} \tag{13}$$

and

$$\vec{A}_2(r,\theta,\varphi,0) = \tfrac{1}{2}(3a/r_0)^{1/2} g r \sin\theta\, e^{-u/2}\hat{\varphi}, \tag{14}$$

where $\hat{\varphi}$ is the third unit vector in the triple $(\hat{r}, \hat{\theta}, \hat{\varphi})$, then Eq. (12) becomes

$$\nabla\cdot\vec{E}_3 = g^2\rho P_2, \tag{15}$$

where $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ is the second Legendre polynomial. Since the right-hand side of this equation is independent of φ, $\vec{E}_3$ will be perpendicular to $\hat{\varphi}$ and thus $\vec{A}_2\cdot\vec{E}_3$ will vanish. By construction (13), the divergence of $\vec{E}_1$ is zero. Thus Eq. (11) is satisfied identically. If the substitutions $\vec{E}_3 = \nabla\psi$ and $\psi(r,\theta) = F(r)P_2(\cos\theta)$ are made, then Eq. (15) becomes

$$F'' + 2r^{-1}F' - 6r^{-2}F = g^2 a r^2 e^{-u}, \tag{16}$$

where the formula (7) was used for ρ. The solution of this equation that is regular at both $r = 0$ and at $r = \infty$ is

$$F(r) = -144 g^2 a r_0^{\ 7} r^{-3}[1 - e^{-u}(1 + u + u^2/2 + u^3/6 + u^4/24 + u^5/144)]. \tag{17}$$

The apparent singularity at $r = 0$ is fictitious. The function F decreases as r^{-3} at large r, and so $\vec{E}_3$ falls off as r^{-4}. The charge Z is therefore completely screened.

The energy of this screened configuration is

$$H_s = (2g^2)^{-1}\int d^3x\,[\vec{E}_3^{\,2} + \vec{E}_1^{\,2} + (\nabla\times\vec{A}_2)^2], \tag{18}$$

and has the value

$$H_s = \frac{Z}{2r_0}\left(1 + \frac{9}{2560}\frac{g^2Z}{4\pi}\right). \tag{19}$$

A comparison of this result with the expression (10) shows that the energy H_s of the screened configuration is less than that of the Coulomb configuration when the color charge exceeds

$$\frac{g^2Z}{4\pi} \geq \frac{320}{57}, \tag{20}$$

which is about 5.61.

The fields of this example do not constitute the configuration of lowest energy, and the solution of which they are the initial conditions is not static.

Noted added.—While this paper was being refereed, an interesting article on color screening by Sikivie and Weiss appeared.[2] The mechanism of color screening is more complicated when the source of color is a dynamical field rather than a prescribed external current. In that case the analysis of this paper as well as that of Sikivie and Weiss must be extended so as to take into account pair creation, gauge rotations of the quark field, and the kinetic energy of the quark field which depends on its covariant derivatives.

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[1]J. Mandula, Phys. Lett. 67B, 175 (1977).

[2]P. Sikivie and N. Weiss, Phys. Rev. Lett. 40, 1411 (1978).

Exact Monopole Solutions in SU(n) Gauge Theory

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We construct exact monopole and dyon solutions in a renormalizable SU(N+1) gauge theory broken down to SU(N)$\otimes$U(1) by one Higgs multiplet in the adjoint representation. The solutions saturate the Bogomolny lower bound and are spherically symmetric with respect to the angular momentum operator $\vec{J}=-i\vec{r}\times\nabla+\vec{T}$, where $\vec{T}$ spans the maximal SO(3) subalgebra of SU(N+1). The solutions are obtained after a remarkable factorization of the relevant coupled *nonlinear* second-order radial equations into a product of coupled *linear* first-order equations.

The work on magnetic monopoles in spontaneously broken gauge theories, initiated by 't Hooft and Polyakov[1] has led to the prediction of stable magnetically charged particles in most unified models of the fundamental interactions. Significant progress has been made through the introduction of topological considerations, which make it possible to classify the allowed monopole solutions in gauge theories. This leaves as an important unresolved aspect the existence and construction of explicit nonsingular solutions to the nonlinear field equations. We have addressed ourselves to this last question and will show that because of a factorization of the relevant coupled *nonlinear* second-order radial equations into a product of *linear* first-order equations, it is indeed possible to construct analytic monopole solutions. This factorization property seems to be one of the most intriguing aspects of our solutions.

Consider a renormalizable theory with one Higgs field $\Phi=\Phi^a\lambda_a$ in the adjoint representation. For a purely magnetic, time-independent solution[2] the Hamiltonian density is

$$\mathcal{H}=(16\pi)^{-1}\{\mathrm{Tr}(\vec{B}+\vec{D}\Phi)^2\mp 2\,\mathrm{Tr}(\vec{B}\circ\vec{D}\Phi)+\eta V(\Phi)\}.$$

It is easy to construct a Higgs potential $V(\Phi)$ that gives rise to a vacuum expectation value, Φ_0, which breaks SU(N+1) down to SU(N)$\otimes$U(1) as is one of the two possibilities.[3] The topologically conserved charge is associated with the U(1) factor of the residual symmetry. We are interested in the Bogomolny-Prasad-Sommerfield limit[4,5] $\eta\to 0$, in which the symmetry breaking remains but the scalar particles become massless. In this case any solution to the first-order partial differential equations $\vec{B}=\pm\vec{D}\Phi$ will saturate the lower bound on the energy, which is directly related to the topological charge:

$$E=\mp\lim_{r\to\infty}[r^2\textstyle\int(d\Omega/8\pi)\mathrm{Tr}(B_r\Phi)].$$

We look for solutions which are spherically symmetric with respect to $\vec{J}=-i\vec{r}\times\nabla+\vec{T}$, where $\vec{T}$ is a representation of some SO(3) subalgebra of SU(N+1). To obtain the radial equations we adopt the formalism of Wilkinson and Goldhaber,[6] who show that the general *Ansatz* may be taken as

$$e\vec{A}=[\vec{M}(r,\hat{r})-\vec{T}]\times\hat{r}/r,\quad \Phi=\Phi(r,\hat{r}),$$

where $\vec{M}$ and Φ are unknown matrix functions. The quantities $\vec{B}$ and $\vec{D}\Phi$ depend only on derivatives with respect to r and t because the angular derivatives in $\nabla=\hat{r}(\partial/\partial r)+\hat{r}\times\hat{r}\times\nabla$ combine with the $\hat{r}\times\vec{T}$ term of the *Ansatz* to yield commutators

