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Are the Stationary, Axially Symmetric Einstein Equations Completely Integrable?

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A linear eigenvalue problem in the spirit of Lax is constructed for the stationary, axially symmetric Einstein equations. It is conjectured that this entails the complete integrability of the system.

Up to now nobody has found a systematic way to solve Einstein's equations even in the stationary, axially symmetric case. Recently Geroch[1] and Kinnersley[2] have discussed a remarkable infinite-parameter group of transformations leaving the equations invariant. This shows that the system of nonlinear partial differential equations is a very special one. In fact I conjecture that they constitute a "completely integrable Hamiltonian system" similar to the well-known sine-Gordon equation (cf. Scott, Chu, and McLaughlin[3]). This conjecture is based on the existence of a linear eigenvalue problem in the spirit of Lax (cf. Ref. 3) with the nonlinear partial differential equations as compatibility equations.

Under the assumption of stationarity and axial symmetry the metric may be reduced to the form[2]

$$ds^2 = ds_1^{\,2} - ds_2^{\,2},$$

with

$$ds_1^{\,2} = \lambda_{ik}dx^i dx^k, \quad i,k = 1,2, \qquad ds_2^{\,2} = h\delta_{ab}dx^a dx^b, \quad a,b = 3,4, \tag{1}$$

where λ_{ik} and h are only functions of x^3 and x^4. As a result of Einstein's equations $R_{\mu\nu}=0$ the symmetric 2×2 matrix λ obeys ($\tau^2 \equiv -\det\lambda > 0$)

$$\partial^a(\tau\lambda^{-1}\partial_a\lambda) = 0, \quad a = 3,4. \tag{2}$$

Equation (2) is to be supplemented by some equation for h, which may, however, be easily integrated once λ is known[4] and will therefore be disregarded in the following.

Introducing the traceless matrix μ given [with $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$] by

$$\mu \equiv i\,\tau^{-1}\sigma_2\lambda, \tag{3}$$

and complex coordinates

$$\xi = x^3 + ix^4, \quad \bar{\xi} = x^3 - ix^4, \tag{4}$$

Eq. (2) becomes (with $\mu_\xi \equiv \partial\mu/\partial\xi$, etc.)

$$(\tau\mu\mu_\xi)_{\bar{\xi}} + (\tau\mu\mu_{\bar{\xi}})_\xi = 0, \tag{5a}$$

to be supplemented by

$$\tau_{\xi\bar{\xi}} = 0. \tag{5b}$$

Equation (5b) means that τ may be taken as the real part of a holomorphic function. Let σ be the corresponding imaginary part defined through $\sigma_\xi = i\tau_\xi$, $\sigma_{\bar{\xi}} = -i\tau_{\bar{\xi}}$. Since $(\tau - i\sigma)_{\bar{\xi}} = 0$ the functions τ and $-\sigma$ may be used as coordinates x^3 and x^4.

Any traceless 2×2 matrix κ may be expanded as

$$\kappa = \sum_{i=1}^{3} k^i Q_i,$$

with $Q_1 = i\sigma_2$, $Q_2 = \sigma_1$, $Q_3 = \sigma_3$ constituting a basis for the Lie algebra sl(2,R). The linear space of the k's corresponding to the linear space sl(2,R) can be equipped with the SL(2,R)-invariant metric

$$k^2 = k \cdot k \equiv \tfrac{1}{2}\operatorname{Tr}\kappa^2 = -(k^1)^2 + (k^2)^2 + (k^3)^2. \qquad (6)$$

In particular

$$\mu = \sum_{i=1}^{3} q^i Q_i ,$$

with $q^2 = 1$ because $\mu^2 = -\tau^{-2}\sigma_2\lambda\sigma_2\lambda = -\tau^{-2}\det\lambda = 1$.

By choosing the square root appropriately the invariants $A \equiv \sqrt{q_\xi{}^2}$ and $\overline{A} \equiv \sqrt{q_{\overline{\xi}}{}^2}$ are complex conjugates of each other. The angle α defined through

$$\cos\alpha \equiv q_\xi \cdot q_{\overline{\xi}}/|A|^2 \qquad (7)$$

turns out to be real.

As a consequence of Eq. (5) the functions A, $\overline{A}$, and α obey the equations

$$A_{\overline{\xi}} + \tfrac{1}{2}\tau^{-1}\tau_\xi \overline{A}\cos\alpha + \tfrac{1}{2}\tau^{-1}\tau_{\overline{\xi}} A = 0, \qquad (8a)$$

$$\overline{A}_\xi + \tfrac{1}{2}\tau^{-1}\tau_{\overline{\xi}} A\cos\alpha + \tfrac{1}{2}\tau^{-1}\tau_\xi \overline{A} = 0, \qquad (8b)$$

$$\alpha_{\xi\overline{\xi}} + |A|^2\sin\alpha - \operatorname{Re}[\tau^{-1}\tau_\xi(\overline{A}/A)\sin\alpha]_\xi = 0. \qquad (8c)$$

For $\tau = A = 1$, Eq. (8c) reduces to the Euclidean sing-Gordon equation, and Eqs. (8a) and (8b) become trivial in that case. The Eqs. (8) are equivalent to Eq. (5a) for μ.

Let ψ be a two-component SU(1,1) spinor, normalized to $\psi^\dagger\sigma_2\psi = 1$, then the following linear equations yield Eqs. (8) as compatibility condition

$$\psi_\xi = \frac{1}{2}\begin{pmatrix} [-\tfrac{1}{2}i\alpha_\xi + \tfrac{1}{2}i\tau^{-1}\tau_{\overline{\xi}}(A/\overline{A})\sin\alpha, & -\overline{\gamma}(s)Ae^{i\alpha/2}] \\ [\overline{\gamma}(s)Ae^{-i\alpha/2}, & \tfrac{1}{2}i\alpha_\xi - \tfrac{1}{2}i\tau^{-1}\tau_{\overline{\xi}}(A/\overline{A})\sin\alpha] \end{pmatrix}\psi \equiv c\psi, \qquad (9a)$$

$$\psi_{\overline{\xi}} = \overline{c}\psi, \qquad (9b)$$

where $\gamma(s)$ is the function

$$\gamma(s) = \left(\frac{1-2s(\sigma+i\tau)}{1-2s(\sigma-i\tau)}\right)^{1/2} = \overline{\gamma}(s)^{-1}, \quad s \in R. \qquad (10)$$

Equations (9a) and (9b) represent a linear "eigenvalue problem" (with $s \in R$ as the "eigenvalue") in the spirit of Lax (cf. Ref. 3). For $\tau = A = 1$ these equations reduce to the ones known for the sine-Gordon equation[5] (or rather a Euclidean version thereof).

The asymptotic behavior of c corresponding to asymptotically Minkowskian solutions of Einstein's equations is given by

$$c_{\text{as}} = \frac{\overline{\gamma}_{\text{as}}(s)}{4\rho}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_{\text{as}}(s) = \left(\frac{1-2is(\rho+iz)}{1+2is(\rho-iz)}\right)^{1/2}, \qquad (11)$$

employing cylindrical coordinates for the Minkowski metric $ds^2 = dt^2 - \rho^2 d\varphi^2 - d\rho^2 - dz^2$.

This shows that the Eqs. (9) differ in two respects from the standard form of the Lax equations (cf. Refs. 3 and 5). Firstly the dependence on the eigenvalue s is rather more involved. Secondly the asymptotic behavior of c does not support freely traveling waves as asymptotic solutions for ψ. Yet c_{as} is still simple enough to nourish some hope that a method similar to the "inverse-scattering method" (cf. Ref. 3) may be developed to reduce the solution of the nonlinear equations (8) to a sequence of linear problems.

A detailed account of the derivation of Eqs. (9) will be given elsewhere.

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