liquid T^{-2} behavior. This deviation is several times larger than the expected fluctuation effect, and does not appear to be related to the superfluid transition. As a result, one cannot make a reliable background subtraction to isolate the effects of superfluid fluctuations.[8]

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Application of the Real-Space Renormalization Group to Dynamic Critical Phenomena

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We present a general approach for applying real-space renormalization-group methods to dynamic critical phenomena. In particular, we discuss the two-dimensional kinetic Ising model treating the interaction between blocks of spins as a small parameter.

We present here a general formulation for applying the real-space renormalization-group (RG) method[1] to critical dynamics. While the analysis will be in terms of two-dimensional kinetic Ising (KI) models,[2,3] we believe the approach to be applicable to higher dimensions and models with order parameters with continuous symmetries.

Consider a set of spins $\{\sigma\}$ on a two-dimensional triangular lattice. We assume that these spins interact via the nearest-neighbor Ising Hamiltonian

$$-H[\sigma]=K\sum_{\langle i,j\rangle}\sigma_i\sigma_j,$$

and the equilibrium probability of having a particular spin configuration $\{\sigma\}$ is given by $P[\sigma]=e^{H[\sigma]}/Z$, where Z is the partition function. In dynamical problems we are interested in the time-correlation functions that can be constructed from the operator

$$G_{\sigma,\sigma'}(t)=\exp(D_\sigma t)\delta_{\sigma,\sigma'}P[\sigma], \qquad (1)$$

where $\delta_{\sigma,\sigma'}$ sets $\sigma_i=\sigma_i'$ on all lattice sites, t is time, and $-iD_\sigma$ is the Liouville operator constructed from H for fully microscopic models. If we are dealing with an Ising model, then D_σ must be constructed by hand. We can construct, for example, the spin-spin correlation function from $G_{\sigma,\sigma'}(t)$ by multiplying by σ_i and σ_j' and summing over all σ and σ'.

In this paper we will study the KI model.[2,3] In this case the two primary properties satisfied by D_σ (or $D[\sigma|\sigma']$ in matrix notation) are

$$\mathrm{Tr}^{\sigma'}D[\sigma|\sigma']P[\sigma']=0 \qquad (2a)$$

(where $\mathrm{Tr}^{\sigma'}$ means to sum over all spins σ') to

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insure that the system decays to the equilibrium state characterized by $P[\sigma]$, and the symmetry relation

$$D[\sigma|\sigma']P[\sigma'] = P[\sigma]D[\sigma'|\sigma]. \tag{2b}$$

We also introduce the adjoint operator $\tilde{D}[\sigma|\sigma'] = D[\sigma'|\sigma]$. The precise form of D that we analyze is

$$\tilde{D}[\sigma|\sigma'] = \sum_{l=1}^{3} \tilde{D}_l[\sigma|\sigma']; \tag{3a}$$

$$\tilde{D}_1[\sigma|\sigma'] = -\textstyle\sum_i \frac{1}{2}\alpha\sigma_i\sigma_i'\Lambda_{\sigma,\sigma'}{}^{[i]}W[\sigma_i], \tag{3b}$$

$$\tilde{D}_2[\sigma|\sigma'] = -\sum_{\langle i,j\rangle} \tfrac{1}{4}\beta\sigma_i\sigma_j'\Lambda_{\sigma,\sigma'}{}^{[i,j]}W[\sigma_i,\sigma_j], \tag{3c}$$

$$\tilde{D}_3[\sigma|\sigma'] = -\sum_{\langle i,j\rangle} \tfrac{9}{4}\sigma_i\sigma_j\sigma_i'\sigma_j'\Lambda_{\sigma,\sigma'}{}^{[i,j,]}W[\sigma_i,\sigma_j], \tag{3d}$$

where $\Lambda_{\sigma,\sigma'}{}^{[i,j,\ldots]}$ sets $\sigma_k = \sigma_k'$ except for the spins at sites i, j, etc., $W[\sigma_i,\sigma_j\ldots]$ is defined such that $W[\sigma_i,\sigma_j,\ldots]P[\sigma]$ is independent of the spins at sites i, j, etc.

In static critical phenomena we implement the real-space RG by introducing an operator $T[\mu|\sigma]$ which maps a group of clustered spins σ onto a new block spin μ. If we demand that $\mathrm{Tr}^{\mu}T[\mu|\sigma] = 1$ then we obtain a probability distribution for the block spins

$$P[\mu] = \mathrm{Tr}^{\sigma}T[\mu|\sigma]P[\sigma]. \tag{4}$$

We can rewrite this as a transformation between Hamiltonians, $H[\sigma] \to H[\mu]$. The dynamical generalization of this procedure corresponds to finding a mapping from D_σ onto the dynamical operator D_μ which governs the dynamics of the block-spin variables. We start by proposing that the appropriate generalization of $G_{\sigma\sigma'}(t)$ corresponding to the μ spins is

$$G_{\mu\mu'}(t) = \mathrm{Tr}^{\sigma}\mathrm{Tr}^{\sigma'}T[\mu|\sigma]T[\mu'|\sigma']G_{\sigma\sigma'}(t) = \mathrm{Tr}^{\sigma}T[\mu|\sigma]\exp(D_\sigma t)T[\mu'|\sigma]P[\sigma]. \tag{5}$$

In this case we "renormalize" the σ and σ' spins separately. Initially

$$G_{\mu\mu'}(t=0) \equiv \tilde{G}_{\mu\mu'} = \mathrm{Tr}^{\sigma}T[\mu|\sigma]T[\mu'|\sigma]P[\sigma]. \tag{6}$$

If we follow Niemeijer and van Leeuwen[1] (NvL) and choose T according to the "majority rule" [Eq. (13) below with $f=1$], then

$$T[\mu|\sigma]T[\mu'|\sigma] = \delta_{\mu,\mu'}T[\mu|\sigma] \tag{7}$$

and

$$\tilde{G}_{\mu\mu'} = \delta_{\mu,\mu'}P[\mu]. \tag{8}$$

More generally we assume that the T's can be chosen such that (8) holds and the corresponding $P[\mu]$ is interpreted as an appropriate probability distribution.

As a first step in obtaining the mapping $D_\sigma \to D_\mu$ we introduce the Laplace transform

$$G_{\mu\mu'}(z) = -i\int_0^{+\infty} dt\, e^{+izt}G_{\mu\mu'}(t) = \mathrm{Tr}^{\sigma}T[\mu|\sigma]R(z)T[\mu'|\sigma]P[\sigma], \tag{9}$$

where $R(z) = [Z - iD_\sigma]^{-1}$ is the resolvent operator. One can then show[4] that $G_{\mu\mu'}(z)$ satisfies the equation

$$\mathrm{Tr}^{\bar{\mu}}(Z\delta_{\mu,\bar{\mu}} - D[\mu|\bar{\mu}])G_{\bar{\mu}\mu'}(z) = \tilde{G}_{\mu\mu'}, \tag{10}$$

where the matrix $D[\mu|\mu']$ is given explicitly by

$$D[\mu|\mu'] = D^s[\mu|\mu'] + D^c[\mu|\mu'], \tag{11a}$$

$$D^s[\mu|\mu']P[\mu'] = \langle T[\mu|\sigma](\tilde{D}_\sigma T[\mu'|\sigma])\rangle, \tag{11b}$$

$$D^c[\mu|\mu']P[\mu'] = -\langle\!\langle(\tilde{D}_\sigma T[\mu|\sigma])R(z)(\tilde{D}_\sigma T[\mu'|\sigma])\rangle\!\rangle + \mathrm{Tr}^{\bar{\mu}}\mathrm{Tr}^{\bar{\mu}'}\langle(\tilde{D}_\sigma T[\mu|\sigma])R(z)T[\bar{\mu}|\sigma]\rangle G_{\bar{\mu}\bar{\mu}'}{}^{-1}(z) \times\langle T[\bar{\mu}'|\sigma]R(z)(\tilde{D}_\sigma T[\mu'|\sigma])\rangle\!\rangle, \tag{11c}$$

where $\langle\rangle$ indicates an average over $P[\sigma]$. One can show directly that $D[\mu|\mu']$ satisfies (2). Thus $D[\mu|\mu']$ satisfies the important property that it is compatible with the new equilibrium probability distribution $P[\mu]$. We offer $D[\mu|\mu']$ as defined by (11) as the appropriate dynamical operator for the μ lattice.

$D[\mu|\mu']$ is frequency dependent through the term D^c. An important question from a RG point of view is whether $D[\mu|\mu']$ can be well approximated by a Markovian or frequency-independent operator. Physically we expect that $D[\mu|\mu']$ will be Markovian (or approximately so) only if the σ degrees of freedom mapped onto the μ spins are dynamically slower than the rest of the σ degrees of freedom. This means that all of the σ degrees of freedom we average over must decay to zero before the μ spins move if we are to have a near-Markovian $D[\mu|\mu']$. The degree to which $D[\mu|\mu']$ is Markovian will depend *strongly* on the choice of $T[\mu|\sigma]$.

We now want to compute $D[\mu|\mu']$ in a perturbation expansion analogous to the static cumulant expansion of NvL.[1] Thus we break the lattice up into cells with three spins per cell. We then treat all intercell couplings as being $O(\epsilon)$. If we start with the usual KI model [(3) with $\beta=q=0$] we find that the q and β terms are generated (with other terms) at $O(\epsilon^2)$ and turn out to have a major effect on the fixed-point structure of the RG recursion relations. Thus we keep the β and q terms from the beginning in our analysis. The other terms generated at $O(\epsilon^2)$ do not appear to be relevant. In the calculation β and q terms are broken up into intracell and intercell pieces and the intercell pieces are treated as being $O(\epsilon)$. The first step in the calculation is to choose the appropriate $T[\mu|\sigma]$. Assume that $T[\mu|\sigma]$ can be written as a product of factors for each cell, where, for the ith cell,

$$T^i[\mu|\sigma]=\tfrac{1}{2}[1+\mu_i\varphi_i(\sigma)], \tag{12}$$

and μ_i is the block spin of the ith cell. If φ_i is symmetric under interchange and odd under flipping of the three spin in the ith cell, we can write

$$\varphi(\sigma)=N(\sigma_T-f\sigma_c), \tag{13}$$

where σ_T is the sum of the three spins in a cell and σ_c is the product of the same three spins. While the "majority rule" choice for T [$f=1$, $N=\frac{1}{2}$ in (13)] may seem most natural, it is not a very good choice from a dynamical point of view. The reason why it is poor is that it mixes slow degrees of freedom (to flip the sign of σ_T requires flipping the sign of three spins) with fast degrees of freedom (to flip the sign of σ_c only requires flipping the sign of any single spin in the cell). A clean separation between fast and slow cell variables can be accomplished by first solving the eigenvalue problem

$$\textstyle\sum_{\sigma'}\tilde{D}^0[\sigma|\sigma']\psi_i(\sigma')=-\lambda_i\psi_i(\sigma), \tag{14}$$

for each cell, where $\tilde{D}^0$ is the dynamical operator [of $O(\epsilon^0)$] corresponding to a particular three-spin cell, and then choosing $\varphi(\sigma)$ in $\boldsymbol{T}$ to be the lowest nontrivial ($\lambda_i\neq 0$) eigenfunction. The resulting eight orthonormal eigenfunctions can be ordered with increasing eigenvalue λ_i. $\lambda_0=0$ corresponds to $\psi_0=1$ and is the smallest eigenvalue. The next smallest eigenvalue is associated with an eigenfunction of the same form as (13) but with N and f depending on K, α, β, and q.

With this new choice for T we are projecting onto the two slowest modes (λ_0 and λ_1) within a cell. We can then compute $D[\mu|\mu']$ in an expansion in the intercell coupling. We find, including terms $O(\epsilon)$, that $\tilde{D}[\mu|\mu']$ has *exactly* the same form as $\tilde{D}[\sigma|\sigma']$ (it is Markovian) but with the new parameters K',α',β',q' given by the recursion relations $x'=\Delta_x x$, $x=\{K,\beta,q\}$, $\Delta_x=2\langle A_x\rangle_0^2$, where $\langle\rangle_0$ indicates an average over $P[\sigma]$ for a single cell, $A_K=\sigma_1\psi_1(\sigma)$, $A_\beta=A_K W_0[\sigma_1]A_q=\psi_1^2(\sigma)\times W_0[\sigma_1]$, and $\Delta_\alpha=\lambda_1/\alpha$ where σ_1 is a spin in a cell and $W_0[\sigma_i]$ is the single-cell component of $W[\sigma_i]$ introduced in (3).

These results are correct to $O(\epsilon)$. Non-Markovian behavior[5] appears only in $O(\epsilon^2)$. The overall time decay of the inverse Laplace transform of D^c is governed by the eigenvalues $\lambda_i>\lambda_1$. This decay modulates the evolution of the μ variables (which decay with the rate λ_1). To the degree that the ratio λ_1/λ_i is small[6] one can also ignore any non-Markovian effects in $O(\epsilon^2)$.

In Table I we show the results of an analysis of the recursion relation discussed above. We have listed only the fixed-point solutions corresponding to $0<K^*<\infty$ and introduced the ratios $R_\beta=\beta/\alpha$ and $R_q=q/\alpha$. The main points of interest are as follows: (i) The $R_\beta^*=0$, $R_q^*\to\infty$ fixed point is the stable fixed point. (ii) The parameter f that appears in the eigenfunction ψ_1 scales to zero near the stable fixed point. (iii) The dynamical critical index z associated with the slowest mode (the β variable) defined by $z_\beta=-\ln\Delta/\ln b$ (where $b=\sqrt{3}$ for a triangular lattice in two dimensions) is relatively insensitive to the particular fixed-point values of α, β, and q. The value $z=1.67$ is reasonable when compared to high-temperature–ex-

TABLE I. Fixed-point values for K, R_β, and R_q and associated critical indices.

K^*	$R_\beta{}^*$	$R_q{}^*$	z_α	z_β	z_q	ν
0.212	0	0	0.840	1.678	−0.185	1.753
0.213	−0.186	0	1.703	1.703	−0.323	1.778
0.212	0	∞	0.836	1.672	−0.064	2.239
0.212	−0.605	∞	1.672	1.672	−0.064	2.239

pansion[7] results $z \simeq 2$ and the rigorous[3] lower bound $z = 1.75$.

The RG dynamics drives ψ_1 to a fixed-point form proportional to σ_T and eliminates the σ_c component. This ties in with our physical argument above that σ_c should be associated with rapidly varying degrees of freedom. We can, in this case, understand why our static results are poor (as given by the exponent ν in the table). The "majority rule" choice with $f=1$ leads to the "optimal" first-order statics.[8] Clearly the fixed-point value $f=0$ will lead to an inferior treatment for the statics.

Finally, we note that there exist other schemes[9] for applying the real-space RG to dynamics. Neither of these methods addresses the question of whether the dynamical operator D_σ approaches a fixed-point form under iteration of the RG.

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