PHYSICAL REVIEW LETTERS

VOLUME 41 | 14 AUGUST 1978 | NUMBER 7

New Exact Solutions of the Classical Sine-Gordon Equation in 2+1 and 3+1 Dimensions

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(Received 22 December 1977; revised manuscript received 16 June 1978)

The method of Bäcklund transformations is employed to derive in 2 + 1 and 3 + 1 dimensions exact solutions of the sine-Gordon equation $[\nabla^2 - c^{-2}(\partial^2/\partial t^2)]\chi = \sin\chi$. A formula is developed in 3 + 1 dimensions which permits us to generate without additional quadratures an infinite class of new time-dependent solutions.

The solution of nonlinear second-order partial differential equations in four dimensions continues to be one of the most tenacious problems in mathematical physics. The recent emphasis among physicists on nonlinear dispersive phenomena[1]—especially in connection with soliton theory[2]—and development of nonperturbative methods[3] in two dimensions have made it even more desirable to find four-dimensional solutions of such prominent equations as the nonlinear Schrödinger equation[4,5] and the ubiquitous sine-Gordon equation.[6-8]

The purpose of this Letter is to report explicit solutions of the classical sine-Gordon equation (SGE) in 2 + 1 and 3 + 1 dimensions, respectively:

$$(\partial_x^2 + \partial_y^2 - \partial_t^2)\chi = \sin\chi, \quad c = 1, \tag{1a}$$

$$(\nabla^2 - \partial_t^2)\chi = \sin\chi, \quad \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad c = 1, \tag{1b}$$

where χ is a scalar field, x, y, z are space variables, t denotes time, and $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, etc. The Bäcklund transformations[9] associated with (1a) and (1b) read, respectively,

$$(I\partial_x + i\sigma_1\partial_y + \sigma_2\partial_t)(\alpha - i\beta)/2 = \sin[(\alpha + i\beta)/2]\exp[i\theta\sigma_1\exp(\lambda\sigma_3)], \tag{2a}$$

$$(I\partial_x + i\sigma_1\partial_y + i\sigma_3\partial_z + \sigma_2\partial_t)(\alpha - i\beta)/2 = \sin[(\alpha + i\beta)/2]\exp\{i\theta\sigma_1\exp[(-i\varphi\sigma_2)\exp(-\tau\sigma_1)]\}, \tag{2b}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices and I is the 2×2 identity matrix. The parameters $\theta, \lambda, \varphi, \tau$, with $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$, $-\infty < \lambda < +\infty$, $-\infty < \tau < +\infty$, are called Bäcklund transformation parameters, while the real functions α, β satisfy

$$(\partial_x^2 + \partial_y^2 - \partial_t^2)\begin{Bmatrix}\alpha(x,y,t)\\ \beta(x,y,t)\end{Bmatrix} = \begin{Bmatrix}\sin\alpha(x,y,t)\\ \sinh\beta(x,y,t)\end{Bmatrix}, \tag{3a, 3b}$$

or

$$(\nabla^2 - \partial_t^2)\begin{Bmatrix}\alpha(x,y,z,t)\\ \beta(x,y,z,t)\end{Bmatrix} = \begin{Bmatrix}\sin\alpha(x,y,z,t)\\ \sinh\beta(x,y,z,t)\end{Bmatrix}. \tag{4a, 4b}$$

Equations (2a) and (2b) imply a transformation from the "old" solution α to the "new" solution $i\beta$:

$$i\beta = B(\theta,\lambda)\alpha, \tag{5a}$$

$$i\beta = B(\theta,\varphi,\tau)\alpha, \tag{5b}$$

with B known as the Bäcklund transformation operator. Equations (5) may be represented symbolically by a Bianchi diagram (see Fig. 1).

The next task is to derive exact solutions for both α and β: I illustrate[7] the procedure for the (2+1)-dimensional SGE (1a). In order to solve (2a), I first replace it by the real equations

$$I\partial_x(\alpha/2) + P(\beta/2) = I\cos\theta\sin(\alpha/2)\cosh(\beta/2) - M\sin\theta\cos(\alpha/2)\sinh(\beta/2), \tag{6a}$$

$$P(\alpha/2) - I\partial_x(\beta/2) = I\cos\theta\cos(\alpha/2)\sinh(\beta/2) + M\sin\theta\sin(\alpha/2)\cosh(\beta/2), \tag{6b}$$

where $P = \sigma_1\partial_y - i\sigma_2\partial_t$ and $M = \sigma_1\exp(\lambda\sigma_3)$. To get α solutions, we set $\beta \equiv \beta_0 = 0$ ("vacuum" solution) in (6) and obtain

$$\alpha_1(x,y,t;\theta,\lambda) = 4\tan^{-1}[a_0\exp T(x,y,t;\theta,\lambda)], \tag{7a}$$

$$T = x\cos\theta + \sin\theta(y\cosh\lambda + t\sinh\lambda), \quad a_0 \text{ constant}, \tag{7b}$$

where α_1 satisfies Eq. (3a). The solitonlike nature of this solution can best be inferred from the asymptotic behavior of α_1 in cylindrical coordinates $x = \rho\cos\tau$, $y = \rho\sin\tau$, and $t = t$, with $0 \leq \tau \leq 2\pi$, $0 \leq \rho < +\infty$, $-\infty < t < +\infty$. For $-\infty < \lambda < +\infty$ and fixed $t \equiv t_0$, $|t_0| < +\infty$, for example, the result is (with $a_0 \equiv 1$)

$$\lim_{\rho\to+\infty}\alpha_1(\rho,\tau,t_0;\theta,\lambda) = \begin{cases} 2\pi, & \text{if } F(\tau;\theta,\lambda) > -1, \\ 0, & \text{if } F(\tau;\theta,\lambda) < -1, \end{cases}$$

where $F \equiv \tan\tau\tan\theta\cosh\lambda$. A similar conclusion holds if we fix $\rho = (x^2+y^2)^{1/2}$ at $\rho \equiv \rho_0 < +\infty$ and allow $t \to \pm\infty$.

Moreover, the choice $\lambda = 0$ in Eq. (7) yields

$$\alpha_1(\rho,\tau;\theta) = 4\tan^{-1}\{a_0\exp[\rho\cos(\tau-\theta)]\}, \quad a_0 > 0,$$

which is precisely the solitonlike solution in 2+0 dimensions.[7] Its asymptotic behavior reads

$$\lim_{\rho\to+\infty}\alpha_1(\rho,\tau;\theta) = 2\pi, \text{ if } -\tfrac{1}{2}\pi < \tau - \theta < \tfrac{1}{2}\pi,$$

$$\lim_{\rho\to+\infty}\alpha_1(\rho,\tau;\theta) = 0, \text{ if } \tfrac{1}{2}\pi < \tau - \theta < \tfrac{3}{2}\pi.$$

Similarly we may derive solutions for β by letting $\alpha \equiv \alpha_0 = 0$ ("vacuum" solution) in (6):

$$\beta_1 = \begin{cases} 4\tanh^{-1}(a_1\exp T), & \text{if } T \leq 0, \quad (8a) \\ 4\coth^{-1}(\bar{a}_1\exp T), & \text{if } T > 0, \quad (8b) \end{cases}$$

T being the same as in (7b); $a_1, \bar{a}_1$ are integration constants and β_1 satisfies (3b). Before examining the (3+1)-dimensional case, we observe (i) that α_1, β_1 depend only on the single variable T (this is an exception, however, and does *not* apply to the general multiple solutions, as discussed in the conclusion) and (ii) that the derivation of these solutions is consistent with the integrability conditions.

In 3+1 dimensions, the simplest nontrivial α and β solutions read

$$\alpha_1(x,y,z,t;\theta,\varphi,\tau) = 4\tan^{-1}(c_0\exp R), \tag{9}$$

$$\beta_1(x,y,z,t;\theta,\varphi,\tau) = \begin{cases} 4\tanh^{-1}(c_1\exp R), & \text{if } R \leq 0, \quad (10a) \\ 4\coth^{-1}(c_2\exp R), & \text{if } R > 0, \quad (10b) \end{cases}$$

$$R = x\cos\theta + y\sin\theta\cos\varphi + \sin\theta\sin\varphi[z\cosh\tau + t\sinh\tau]; \tag{11}$$

c_0, c_1, c_2 are integration constants, $(\nabla^2 - \partial_t^2)\alpha_1 = \sin\alpha_1$, while $(\nabla^2 - \partial_t^2)\beta_1 = \sinh\beta_1$.

One of the advantages of possessing a Bäcklund transformation is that it virtually guarantees the existence of a *generating formula* which enables us to derive without additional quadratures other solutions of the same equation. The following

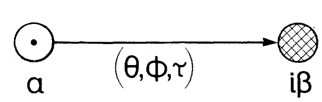

FIG. 1. Bianchi diagram for the Bäcklund transformation [Eq. (2)] which is characterized by the real parameters (θ,φ,τ).

theorem tells us how to generate in 3 + 1 dimensions infinitely many real solutions α. I utilize Fig. 2 and abbreviate $\beta_1(x,y,z,t;\theta_j,\varphi_j,\tau_j) \equiv \beta_1^{(j)}$, $j=1,2$, where $0 \leq \theta_j \leq 2\pi$, $0 \leq \varphi_j \leq 2\pi$, and $-\infty < \tau_j < +\infty$.

Theorem: Let α_0 be a solution of Eq. (4b) which are connected by the Bäcklund transformation $i\beta_1^{(j)} = B(\theta_j,\varphi_j,\tau_j)\alpha_0$. A new solution $\alpha_2(x,y,z,t;\theta_1,\theta_2,\varphi_1,\varphi_2,\tau_1,\tau_2)$ is then given by

$$\tan[(\alpha_2-\alpha_0)/4] = D\tanh[(\beta_1^{(1)}-\beta_1^{(2)})/4], \tag{12a}$$

$$D(\theta_1,\theta_2,\varphi_1,\varphi_2,\tau_1,\tau_2) = \pm[(1+L)/(1-L)]^{1/2}, \quad 1-L \neq 0, \tag{12b}$$

$$L = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2[\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2\cosh(\tau_1-\tau_2)]. \tag{12c}$$

Proof: From Fig. 2 I deduce the *four* transformations

$$i\beta_1^{(j)} = B(\theta_j,\varphi_j,\tau_j)\alpha_0, \quad j=1,2, \tag{13a}$$

$$\alpha_2 = B(\theta_{2-k},\varphi_{2-k},\tau_{2-k})i\beta_1^{(1+k)}, \quad k=0,1, \tag{13b}$$

which are equivalent to the first-order equations

$$K(\alpha_0 - i\beta_1^{(j)})/2 = \sin[(\alpha_0+i\beta_1^{(j)})/2]\exp(i\theta_j S_j), \quad j=1,2, \tag{14a}$$

$$K(i\beta_1^{(1+k)} - \alpha_2)/2 = \sin[(\alpha_2+i\beta_1^{(1+k)})/2]\exp(i\theta_{2-k}S_{2-k}), \quad k=0,1, \tag{14b}$$

where $K \equiv I\partial_x + i\sigma_1\partial_y + i\sigma_3\partial_z + \sigma_2\partial_t$ is the differential operator appearing in Eq. (2b), and $S_m = \sigma_1\exp[(-i\varphi_m\sigma_2) \times \exp(-\tau_m\sigma_1)]$, $m=1,2$. Elimination of the operator K from (14) yields, after further manipulations,

$$[\exp(i\theta_1 S_1) - \exp(i\theta_2 S_2)]\tan[(\alpha_2-\alpha_0)/4] = i[\exp(i\theta_1 S_1) + \exp(i\theta_2 S_2)]\tanh[(\beta_1^{(1)}-\beta_1^{(2)})/4], \tag{15}$$

provided $\cos[(\alpha_2+\alpha_0+i\beta_1^{(1)}+i\beta_1^{(2)})/4] \neq 0$. Multiplying Eq. (15) from the left by its complex conjugate, we can solve the resulting *diagonal* matrix equation consistently for $\tan[(\alpha_2-\alpha_0)/4]$ to obtain the generating formula (12). Note that the "given" solution α_0 need *not* be the vacuum solution $\alpha_0 \equiv 0$.

In deriving the above generating formula I assumed the validity of the theorem of permutability which implies, in turn, the existence of the diagram in Fig. 2. Although this theorem has, as yet, not been proven rigorously for 3 + 1 dimensions, it would appear that the commutative property $\alpha_2 = B_2B_1\alpha_0 = B_1B_2\alpha_0$ of two successive Bäcklund transformations survives, in a certain sense, also in 3 + 1 dimensions [here $B_j \equiv B(\theta_j,\varphi_j,\tau_j)$]. My optimism is based on the fact that α_2 is indeed a solution of (4a), as may be verified by substituting (12a) into $(\nabla^2 - \partial_t^2)\alpha_2 = \sin\alpha_2$. Since the resulting computation is too lengthy for inclusion here (it takes about 15 pages), I shall merely hint at the general procedure. It is convenient to set $\alpha_0 = 0$, express (12a) as $\tan(\alpha_2/4) = D(h_1-h_2)/(1-h_1h_2)$, with $h_j \equiv \tanh(\beta_1^{(j)}/4)$, $j=1,2$, and then calculate $\partial_x^2\alpha_2$, $\partial_y^2\alpha_2$, etc. Combining these second derivatives and simplifying the various expressions, we obtain eventually

$$(\nabla^2 - \partial_t^2)\alpha_2 = M/N, \tag{16}$$

with

$$M = 4D(h_1-h_2)(1-h_1h_2)[(1-h_1h_2)^2 - D^2(h_1-h_2)^2],$$

$$N = [(1-h_1h_2)^2 + D^2(h_1-h_2)^2]^2.$$

In order to compare M/N with $\sin\alpha_2$, we must rewrite $\sin\alpha_2$ as $4[1-\tan^2(\alpha_2/4)][1+\tan^2(\alpha_2/4)]^{-2}\tan(\alpha_2/4)$ and in it replace every $\tan(\alpha_2/4)$ by $D(h_1-h_2)/(1-h_1h_2)$. This exercise yields precisely M/N and verifies that $(\nabla^2-\partial_t^2)\alpha_2 = \sin\alpha_2$.

In conclusion, I should like to make these remarks. (i) The generating formula for α solutions in 2 + 1 dimensions reads the same as (12a), but with L in (12c) replaced by $l = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cosh(\lambda_1-\lambda_2)$. (ii) The α_1 solutions

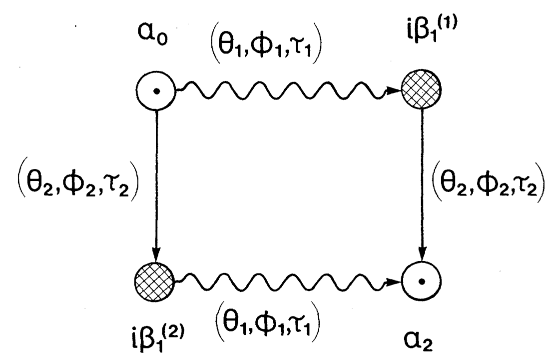

FIG. 2. Bianchi diagram used in the proof of the generating formula (12).

(7) and (9) could also have been obtained without the Bäcklund transformations (2) by subjecting the *one*-soliton solution in $1+1$ dimensions to a Lorentz transformation. The same is true for the α_2 solutions. While neither α_1 nor α_2 is truly a three-dimensional solution of Eq. (3a), as mentioned in connection with Eqs. (7) and (8), it can be shown that there exist an infinite number of multiple solutions α_{2n}, $n=2,3,4,\ldots$, which, for typical values of the Bäcklund parameters (θ,λ), are *genuinely three dimensional.* These, then are *new* solutions, since they *cannot* be derived by a simple rotation from the corresponding multiple solutions in $1+1$ dimensions. (iii) The technique described above enables us to write down new exact solutions of Josephson's equation[7]

$$[\partial_x^2+\partial_y^2-c_0^{-2}(\partial^2/\partial t^2)]\psi(x,y,t) = \lambda_J^{-2}\sin\psi(x,y,t)$$

which describes the propagation of magnetic flux through a Josephson tunneling junction.

It is a pleasure to thank Sidney Coleman for his constructive criticism at various stages of this investigation and Jörg Stehr for some helpful correspondence. I should also like to thank Howard Georgi and Sidney Coleman, as well as the secretarial staff, for their hospitality and assistance during my stay in the Theoretical Physics Group. This research was supported in part by the National Science Foundation under Grant No. 77-22864, and by the National Research Council of Canada, under Grant No. A8063.

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Photon and Pion Emission from the Nucleon-Antinucleon System

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Quantitative estimates of the single-γ and -π emission rates for transitions between bound states of the nucleon-antinucleon system are presented. Quantum-number assignments in the context of potential and baryonium models are suggested for new mesons recently seen in monoenergetic γ emission from the $\bar{p}p$ atom. It is shown that π transitions are important in distinguishing between alternative models.

There has been considerable interest recently in the spectroscopy of mesons whose masses lie in the vicinity of the antinucleon-nucleon ($\bar{N}N$) threshold. Such structures have been seen, for example, in $\bar{N}N$ total and elastic cross sections,[1] $\bar{p}d$ spectator experiments,[2] and πp production experiments,[3] and through the observation of γ rays from the $\bar{p}p$ system.[4]

There exist numerous theoretical predictions for new mesons near the $\bar{N}N$ threshold. Some of these approaches involve the use of $\bar{N}N$ potential models,[5] while others are based on topological expansions[6] or the Massachusetts Institute of Technology bag model extended to the diquark-antidiquark ($Q^2\bar{Q}^2$) sector.[7]

In the present Letter, we adopt the potential approach. We use a model for the intermediate and long-range parts of the strong $\bar{N}N$ potential,

