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Condensed Monopoles and Abelian Confinement

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We show that contribution of monopole loops to the functional integral in four-dimensional lattice (quantum electrodynamics) can be mapped on to scalar quantum electrodynamics. The phase transition where the monopole loops become very long then corresponds to the onset of the Higgs-Kibble mechanism and confinement due to the dual Meissner effect.

There has been a great deal of enthusiasm recently about the possibility that an understanding of the role of extended structures in Yang-Mills theory will reveal the mechanism of quark confinement. Two of the most suggestive ideas appeared in papers by Polyakov,[1] by Banks, Myerson, and Kogut,[2] and by other authors cited in Ref. 2.

Polyakov considered the role of monopoles in a $(2+1)$-dimensional model (where they appear as point objects), and wrote the grand canonical ensemble for these objects as a field theory. He showed that Wilson loops produced strings in the mean-field limit.

Kogut *et al.* considered on a lattice the role of threadlike configurations [e.g., vortex lines in the three-dimensional X-Y model, monopoles in $3+1$ dimensions]. They argued that when the entropy/unit length of such configurations exceeded the action/unit length then they could become very long at no cost: The vacuum is filled with a spaghetti of tangled loops. They were, however, unable to show directly that this phase transition leads to strings. This we demonstrate here.

We combine the ideas of Banks, Myerson, and Kogut and Polyakov with those of various workers in the field of polymer physics[3] by writing down the grand canonical ensemble for interacting loops. We show that this is isomorphic to scalar QED (quantum electrodynamics).

The phase transition to the spaghetti vacuum becomes the transition to the Goldstone mode where the Higgs-Kibble mechanism counteracts the long-range interaction. Evaluating Wilson loops in $(\mathrm{QED})_4$ or spin-spin correlation functions in $(X\text{-}Y)_3$ inserts sources that look like (in the scalar QED language) monopole lines or points, which in the ordered phase have no option but to form Abrikosov vortex lines or sheets. The key problem is to count the number of configurations of a closed loop, which wanders at random through a lattice, in such a way as to make the inclusion of interactions between separate loops, or between different parts of the same loop, straightforward.

The type of quantity which is basic to this is

$$\int_0^\infty P(t)e^{-At}\,dt, \tag{1}$$

where $P(t)$ is the number of closed random configurations of length-t steps. To evaluate this we note that a random walk on a lattice has a probability distribution satisfying

$$\Delta_t\rho = \frac{L^2}{2D}\,\Delta^2\rho;\quad \rho(x,0)=\delta^D(x)\,, \tag{2}$$

where $\rho(x,t)$ is the probability of ending at x after a walk of t steps.

The solution to this can be written, in a continuum approximation, as a Gaussian path integral:

$$\rho(x,t) = N\int_0^x d[x(t)]\,\exp\left\{-\int_0^t \frac{D}{2L^2}\dot{x}^2 dt\right\}. \tag{3}$$

(N is such that $\int \rho d^D x = 1$) and the total number of paths from 0 to x of t steps is

$$\Gamma(x,t) = (2D)^t\rho(x,t) \tag{4}$$

$$= N\int d[x]\exp\left[-\int\left\{\frac{D}{2L^x}\dot{x}^2 - \ln 2D\right\}dt\right]. \tag{5}$$

Now

$$\int_0^\infty \Gamma(x,x,t)e^{-At}dt\,d^Dx = \int dt\,e^{-At}2tP(t)\,, \tag{6}$$

the factor of 2 coming from the fact that a loop from x to x' is indistinguishable from a loop from x' to x and the factor t arising because all t starting ponts on the loop define the same configuration. Writing

$$\Gamma(x,x',t) = \textstyle\sum_n \Phi_n(x)\Phi_n(x')\exp[-\omega_n{}^2(t)],\quad \left\{-\frac{L^2}{2D}\nabla^2 - \ln 2D\right\}\Phi_n = \omega_n{}^2\Phi_n;\ \int\Phi_n{}^2 = 1\,,$$

we see that

$$\int_0^\infty 2tP(t)e^{-At}dt = \sum_n \frac{1}{\omega_n^2 + A} = \mathrm{Tr}\left\{-\frac{L^2\nabla^2}{2D} + A - \ln 2D\right\}^{-1}, \tag{7}$$

so that finally, integrating with respect to A, we find

$$\int_0^\infty P(t)e^{-At}dt = -\tfrac{1}{2}\ln\det\left\{-\frac{L^2}{2D}\nabla^2 + A - \ln 2D\right\}. \tag{8}$$

We have managed to express the statistical sum over closed configurations of action A/unit step in terms of the well known expression for a one-loop Feynman diagram.

If we consider a noninteracting gas of such loops then we have to exponentiate the configuration sum for one loop:

$$e^{-F} = \textstyle\sum_n\{\int_0^\infty P(t)e^{-At}\}^n/n!$$

$$= \det{}^{-1/2}\{-(L^2/2D)\nabla^2 + A - \ln 2D\}$$

$$= \int d[\varphi]\exp[-\int d^Dx\{(L^2/2D)(\nabla\varphi/2)^2 + [A-\ln(2D)](\varphi^2/2)\}. \tag{9}$$

So a gas of noninteracting loops with a certain action/unit can be written as a free-field theory with $m^2 = A - \ln(2D)$.

The path-integral representation can be used to introduce interactions between different elements of the loop. In both the $(X\text{-}Y)_3$ and $(\mathrm{QED})_4$ cases the topological excitations interact with a Biot-Savart-like law.[2] Using the fact that

$$G(x,x',t) = \int_x^{x'} d[x]\exp[-\int_0^t\{\tfrac{1}{2}\dot{x}^2 + V(x) + ieA_\mu\dot{x}_\mu\}dt]$$

satisfies

$$\frac{\partial G}{\partial t} = \tfrac{1}{2}(\partial_\mu + ieA_\mu)^2G - VG\,, \tag{10}$$

we can retrace our previous analysis and include the interaction

$$V_{\rm int}=\tfrac{1}{2}e^2\int dx_1{}^{\mu}G_{\mu\nu}(x_1,x_2)dx_2{}^{\nu} \tag{11}$$

by means of a Gaussian trick to obtain

$$e^{-F}=\sum_n\int d[A_\mu]\left[\int_{\rm loops} d[x]\exp\{-\int(\tfrac{1}{2}\dot{x}^2+m^2+ieA_\mu\dot{x})dt\}\right]^n/n!\,\exp\left[-\tfrac{1}{2}\int A_\mu(x)G_{\mu\nu}(x,y)A_\nu(y)d^Dx\,d^Dy\right] \tag{12}$$

for the grand canonical variable of interacting loops.

This can be written again as

$$e^{-F}=N\int d[A]d[\varphi]\,d[\varphi^*]\,\exp\left[-\int d^Dx\,\{|\nabla\varphi|^2+m^2+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\lambda\varphi^4\}\right]. \tag{13}$$

The pair of complex fields is required because the loops are oriented and the factor of 2 in (6) must vanish. We have introduced a repulsive term to take into account that closed loops annihiate each other or are suppressed for other reasons. This prevents the density of loops being infinite when $A<\ln 2D$ or $m^2<0$. By differentiating with respect to m^2 on can see that $\langle\varphi^*\varphi\rangle$ is the loop density and $\langle\varphi^*\rangle\langle\varphi\rangle$ is that fraction which is of infinite length. Fortunately the correlation inequality

$$\langle\varphi^*\varphi\rangle\geqslant\langle\varphi\rangle\langle\varphi^*\rangle \tag{14}$$

keeps this fraction from being greater than one. We therefore see the connection between the Higgs-Kibble mechanism, operative for $m^2<0$, and the spaghetti vacuum phase of Kogut *et al.*

The effect of adding terms to produce correlation functions or Wilson loops is simply to replace $F_{\mu\nu}$ by $F_{\mu\nu}+G_{\mu\nu}$ where

$$\partial_\mu{}^*G_{\mu\nu}=K_\nu,$$

$$K_\nu=\sum_{x_i}g\delta^3(x-x_i)\quad\text{in }(X\text{-}Y)_3,$$

$$K_\nu=g\oint\frac{dy^\nu}{dt}\delta^4(x-g(t))dt\quad\text{in (QED)}_4.$$

In both cases the charges satisfy the relevant Dirac condition

$$eg=2\pi$$

required for consistency. These sources will produce Nielson-Oleson strings or sheets producing confinement in the $m^2<0$ phase.

The confinement mechanism is therefore a Bose condensate of monopoles as predicted by 't Hooft[4] and Mandelstam.[5] For more details of our reasoning readers are referred to a more extended paper.[6]

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