Kinetic Analysis of the Localized Magnetohydrodynamic Ballooning Mode

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A kinetic analysis of magnetohydrodynamic ballooning modes has been carried out in the limit of large toroidal mode numbers. At marginal stability, the mode acquires a real frequency. The finite ion Larmor radius and perturbed electron pressure anisotropy are stabilizing effects, whereas the perturbed ion pressure anisotropy and the longitudinal electric field are destabilizing. By minimizing the stability effect and maximizing the destabilizing effect, the critical β may be 10% lower than that predicted by the magnetohydrodynamic theory.

Tokamaks achieve stability against pressure-driven magnetohydrodynamic (MHD) instabilities at low pressure through the average magnetic well due to toroidality and through shear in the magnetic lines of force.[1] It has been known for many years that if the plasma pressure exceeds a critical value, ballooning modes[2] could be driven unstable by the localized regions of unfavorable curvature. Roughly speaking, the predicted critical condition is $\gamma\tau_A<1$, where γ is the flute-mode growth rates characteristic of the bad-curvature region, and τ_A is the transit time of a shear Alfvén wave between the regions of good and bad curvature. The critical β depends on the detailed geometry of the device and may be enhanced by suitable choice of cross-sectional shape for the plasma. Recently, some confirmation of the critical conditions for the onset of ballooning modes has been obtained through numerical[3] and analytical[4] studies of the MHD energy principle. However, in fusion-reactor conditions, some of the assumptions for the MHD model may not be satisfied. Therefore, in this article we study the stability of the ballooning modes from a kinetic approach.

To begin our analysis, we follow Rosenbluth and Sloan[5] in developing a variational formulation for perturbations in finite-β plasmas, except that for frequencies ω around the ion diamagnetic frequency $\omega_i^* \ll \Omega_i$ (the ion gyrofrequency) the ion inertia term must be retained. We also extend their analysis to the "flute" regime where $\omega \gg \omega_{bi}$, the ion bounce frequency. The perturbed electric field $\vec{E}_1$ and magnetic field $\vec{B}_1$ are of the form $\exp(i\omega t - in\theta)$. They are related to the scalar potential φ and vector potential $\vec{A}$ by

$$\vec{E}_1 = -\nabla\varphi - \partial\vec{A}/\partial t, \quad \vec{B}_1 = \nabla\times\vec{A}. \tag{1}$$

The vector potential $\vec{A}$ is specified through the displacement $\vec{\xi}$ by

$$\vec{A} = \vec{\xi}\times\vec{B}. \tag{2}$$

Only $\vec{\xi}_\perp$ enters $\vec{A}$; $\xi_\parallel$ can be chosen to vanish.[5] We also require $\vec{\xi}=0$ at the plasma boundary. The metric we use is given by $dl^2 = d\psi^2/R^2B_\chi^{\,2} + J^2B_\chi^{\,2}d\chi^2 + R^2d\theta^2$.

For variations with respect to φ^* and $\vec{\xi}^*$ the variational principle may be written in the form

$$\int K\,d\tau + \int W\,d\tau = 0. \tag{3}$$

Here $d\tau$ is the volume element, K is a modified perpendicular kinetic energy density,

$$K = -\rho|\xi|^2(\omega - \tfrac{1}{2}\omega_i^*)^2, \tag{4}$$

ρ being the mass density, and the electron inertia is neglected. The perturbed potential-energy density may be written as

$$W = W_{\rm MHD} + W_{pi} + W_{pe} + W_{FLR} + W_\varphi, \tag{5}$$

where

$$W_{\rm MHD} = \frac{1}{4\pi}\frac{1}{B^2}|\vec{B}_1\times\vec{B}|^2 + \frac{1}{4\pi}\frac{1}{B^2}(\vec{B}_1\cdot\vec{B} - 4\pi\vec{\xi}\cdot\nabla p)^2 - \frac{\vec{J}\cdot\vec{B}}{B^2}(\vec{\xi}^*\times\vec{B}\cdot\vec{B}_1) - 2(\vec{\xi}\cdot\nabla p)(\vec{\xi}^*\cdot\vec{\kappa}), \tag{6}$$

$$W_{pi} = \frac{(\omega-\omega_i^*)}{T_i}\left\langle\frac{|e\varphi + \mu B_\parallel - e\vec{A}\cdot\vec{u}_{di}|^2}{\omega-\omega_{Di}}\right\rangle, \tag{7}$$

$$W_{pe} = \frac{(\omega-\omega_e^*)}{T_e}\left\langle\frac{1}{\omega-\overline{\omega}_{De}+i\nu}\overline{|e\varphi + \mu B_\parallel + e\vec{A}\circ\vec{u}_{de}|^2}\right\rangle, \tag{8}$$

$$W_{FLR} = \frac{\omega_i^{*2}}{4}\rho|\xi|^2 + \frac{(\omega-\omega_i^*)}{T_i}\left\langle\frac{V_\perp^2}{2\Omega_i^2}\nabla_\perp^2\left(\frac{|e\varphi+\mu B_\parallel - e\vec{A}\cdot\vec{u}_{di}|^2}{\omega-\omega_{Di}}\right)\right\rangle, \tag{9}$$

and

$$W_\varphi = -\frac{e^2|\varphi|^2}{T}p. \tag{10}$$

This is the basic formulation for studying finite-β plasma modes with frequency $\sim\omega_i^*$. In the MHD model, the plasma pressure response is assumed to be adiabatic. Hence the perturbed potential energy contains a $\gamma p(\nabla\cdot\vec{\xi})^2$ term. Aside from the $\gamma p(\nabla\cdot\vec{\xi})^2$ term, $W_{\rm MHD}$ in (6) is identical to that used by Dobrott *et al.*; p, $\vec{J}$, and $\vec{\kappa}$ are the equilibrium pressure, current, and field line curvature. In $W_{\rm MHD}$, the first term is the energy needed in bending the magnetic field line, the second term is the work done in compressing the magnetic field and the plasma, the third term drives the kink instability, and the fourth term, which drives the ballooning and interchange modes, is the interaction of the pressure gradient and the magnetic field line curvature. W_{pi} is due to the ion pressure anisotropy, with ω_{Di} the ion magnetic drift frequency and u_d the magnetic drift velocity; $\langle\rangle$ stands for the average over velocity space. W_{pe} is due to the electron pressure anisotropy; the bar indicates a time average over the trapped-particle orbit. ν is the frequency at which trapped electrons scatter into the transit regions of velocity space. W_{FLR} consists of a part due to the shift in the mode frequency, and a part due to the ion pressure anisotropy. W_φ is the destabilization energy due to the coupling of the longitudinal electric field to the perpendicular displacement $\vec{\xi}_\perp$. For simplicity, in W_{pi} and W_{FLR}, we include here only the expressions for ions in the flute regime, $\omega > \omega_{bi}$. The corresponding expression for ions in the trapped regime $\omega < \omega_{bi}$ is obvious.

The mode structure is dictated by $W_{\rm MHD}$. At marginal stability, $\omega = \frac{1}{2}\omega_i^*$, and $K=0$. It is easy to see that when the pressure gradient length L_n is shorter than the magnetic field gradient length L_B, $W_{pi}<0$, $W_{pe}>0$, and $W_\varphi<0$. Because of the negative nature of the $\nabla_\perp^2$ operator, W_{FLR} is always positive. We note that by taking the limit $m/e\to 0$, and $\varphi\to 0$, we obtain the energy principle given by Connor and Hastie to study the trapped-particle stabilization of localized interchange modes.[6] The stabilizing effect of the trapped particles on the ballooning mode has been noted independently by Rutherford, Chen, and Rosenbluth.[7]

To evaluate the size of the various kinetic modification terms, we obtain from (3) the Euler equation appropriate for high-n ballooning modes. In the high-n ordering $\nabla_\perp\sim n/R\sim 1/\rho_i$ and $\nabla_\parallel\sim 1/R$. Following Dobrott *et al.*,[4] we obtain to the lowest order $\nabla\cdot\vec{\xi}_\perp^{(0)}=0$. Therefore $\xi_\psi=(1/RB_\chi)\partial Y/\partial\theta$ and $\xi_u=-(RB_\chi/B)\partial Y/\partial\psi$; here, Y is a stream function, $\hat{u}$ being in the direction of $\hat{n}\times\hat{\psi}$. In the next order, the kink driving term becomes a complete differential. Minimization with respect to $(\nabla\cdot\vec{\xi}_\perp)^{(1)}$ gives (with the neglect of terms proportional to $\beta L_n/L_B$),

$$(\nabla\cdot\vec{\xi}_\perp)^{(1)} = -2\vec{\xi}_\perp\cdot\vec{\kappa}. \tag{11}$$

We may now vary with respect to φ^* and obtain the Poisson equation written in the form

$$L(\omega)[e\varphi/T] = S(\omega)[\vec{\xi}_\perp\cdot\vec{\kappa}]. \tag{12}$$

$L(\omega)$ and $S(\omega)$ are two operators depending on frequency. At $\omega=\frac{1}{2}\omega_i^*$ they are given by

$$L[f] = 1 + \frac{1}{2}\frac{\omega_i^*}{\omega_i^* - 2\overline{\omega}_{Di}}(1-k_\perp^2\rho_i^2) - \frac{3}{2}\left\langle\frac{\omega_i^*}{\omega_i^*+2\overline{\omega}_{Di}+i\nu}\overline{f}\right\rangle, \tag{13}$$

$$S[f] = -\left\{\frac{\omega_i^*}{\omega_i^*-2\overline{\omega}_{Di}}(1-k_\perp^2\rho_i^2) + \frac{1}{2}\left\langle\frac{3\omega_i^*}{\omega_i^*+2\overline{\omega}_{Di}+i\nu}\left(\overline{\frac{2\epsilon-\mu B}{T}f}\right)\right\rangle\right\}. \tag{14}$$

In (13) and (14), the $\langle\rangle$ is over the trapped electrons only. With the substitution of $L^{-1}S[\vec{\xi}_\perp\cdot\vec{\kappa}]$ for $e\varphi/T$ into (3), the variational is now in terms of only one function, Y. We note that Eq. (12) stands for the response of the longitudinal electric field to a perpendicular perturbation given by $\vec{\xi}_\perp\cdot\vec{\kappa}$. It is a nonlocal operator characterized by both $\vec{k}_\perp$ and the average over the trapped-electron trajectory. Here we make the further assumption that the eigenfunction for $\vec{\xi}_\perp\cdot\vec{\kappa}$ will also be the approximate eigenfunction for both L and S. Therefore both L and S may be approximated by constants. The resultant variation with respect to $X^*=RBp\xi_\psi^*$ gives the Euler equation as

$$\frac{Bk_\parallel}{4\pi}\left\{\frac{B}{R^2B_\chi^2}\left(1+\frac{d^2}{dh^2}\right)k_\parallel X\right\} - \frac{2}{RB_\chi}\frac{dp}{d\psi}\kappa_x X + \frac{\rho}{R^2B_\chi^2}\left(\frac{\omega_i^{*2}}{4}-\lambda\right)\left[1+\frac{d^2}{dh^2}\right]X$$
$$-\frac{p}{R^2B_\chi^2}\left\{\left[\frac{3}{2}\left(\frac{S}{L}\right)^2+2\frac{S}{L}+3\right]\kappa_x^2+g\right\}X=0. \tag{15}$$

In (15)

$$iBk_{\parallel} = \frac{1}{J}\left(\frac{\partial}{\partial\chi} + in\,\frac{B_\theta J}{R}\right), \quad \frac{d}{dh} \equiv -i\,\frac{R^2 B_\chi{}^2}{nB}\,\frac{\partial}{\partial\psi},$$

$\kappa_x = \kappa_\psi - \kappa_u d/dh$, κ_ψ and κ_u being the normal and geodesic curvature, g is the contribution from the trapped-electron response, and $\lambda = (\omega - \frac{1}{2}\omega_i{}^*)^2$. The general characteristics of this type of equation has been discussed by Roberts and Taylor.[8] By using the quasimode transformation given by Connor, Hastie, and Taylor,[8] (15) is reducible to

$$\frac{1}{4\pi}\,\frac{1}{J}\,\frac{d}{dy}\left\{\frac{1}{JR^2B_\chi{}^2}[1+S_q{}^2]\,\frac{dF}{dy}\right\} + \frac{2}{RB_\chi}\,\frac{dp}{d\psi}\kappa_F F - \frac{\rho}{R^2B_\chi{}^2}\left(\frac{\omega_i{}^{*2}}{4} - \lambda\right)[1+S_q{}^2]F + \frac{p}{R^2B_\chi{}^2}\left\{\kappa_F{}^2\left[\left(\frac{S}{L}\right)^2\frac{3}{2} + 2\,\frac{S}{L} + 3\right] + g\right\}F = 0, \quad (16)$$

with

$$S_q = \frac{R^2B_\chi{}^2}{B}\int^y \frac{\partial}{\partial\psi}\left(\frac{JI}{R^2}\right)dy, \quad \kappa_F = \kappa_\psi + \kappa_u S_q.$$

y is the Fourier transform variable[9] for χ. The boundary condition is $F \to 0$ as $y \to \pm\infty$.

To look for the worst mode, we minimize the stabilizing effects. W_{FLR} would be minimized if $k_\perp \rho_i$ is minimized subject to the physical constraints of the system. To minimize the trapped-electron stabilization, we choose $\nu \gg \omega$, and $g = 0$. Next we note that when the ions are trapped, the destabilizing effect due to the coupling to the longitudinal electric field is much reduced. We therefore take the ions in the flute regime. To lowest order in $L_n/L_B = \epsilon \ll 1$, we obtain $L \sim \frac{3}{2}$, $S \sim -1$. For a shear-free case, at marginal stability, Eq. (16) is reduced to

$$\frac{d^2F}{dy^2} + \frac{\beta}{\beta_0}\left(\tfrac{11}{24}\epsilon + \cos y + \epsilon\tfrac{17}{24}\cos 2y\right)F = 0. \quad (17)$$

Here $\beta_0 = \epsilon/q^2$ and $\beta = 8\pi p/B^2$. The two terms proportional to ϵ are due to effects not contained in the ideal, single-fluid MHD theory. In particular, they are due to the combined effect of the coupling to the longitudinal electric field and the finite ion Larmor radius. It is seen that the critical β is reduced from that due to MHD by $\beta_K/\beta_{\text{MHD}} \simeq 1/(1+\alpha\epsilon)$, $\alpha \sim 1$. For a present-day tokamak with $\epsilon \sim \frac{1}{5}$, the reduction could be $\gtrsim 10\%$.

The major differences between the MHD approach and the kinetic theory (KT) approach may be summarized as follows. In MHD theory, the frequency at marginal stability is zero, the perturbed electron and ion pressures are always isotropic, and any coupling to the longitudinal electric field is neglected. In KT, this mode acquires a real frequency at half of the ion diamagnetic frequency. This shift in frequency provides a finite-Larmor-radius stabilization effect.[10] The perturbed pressures are also anisotropic in KT. The electron pressure anisotropy provides a stabilizing effect, whereas the ion pressure anisotropy is destabilizing. Depending on the toroidal mode number n, the ion bounce frequency can be either higher (trapped ion) or lower (flute ion) than the mode frequency. In the flute-ion regime, the destabilizing effect due to the ion pressure anisotropy is larger than the stabilizing effect of the electron pressure anisotropy. In the trapped-ion regime, the stabilizing effect of the electrons is larger. Further, the stabilizing effect of the electrons may be nullified if the mode frequency is much smaller than the trapped-electron collision frequency. In KT, an important destabilizing effect also comes from the coupling to the longitudinal electric field. In a real situation, the worst mode is likely to be the one with the lowest β threshold. This will occur at the long wavelength end of the flute-ion regime. With the effect of the coupling to longitudinal electric field taken into account, the critical β could be up to 10% smaller than that predicted by the MHD theory.

Equation (16) has been integrated numerically from $y = -\infty$ to $y = +\infty$ to find the marginal $\lambda = 0$ and the quantitative value of the critical β for realistic toroidal equilibria (obtained by numerical techniques). For instance, the critical β_K for a possible JET (Joint European Tokamak) configuration is around 11% lower than β_{MHD}. Systematic application of this theory is in progress and will be reported separately.

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Broken Spin-Orbit Symmetry in Superfluid ^{3}He and *B*-Phase Dynamics

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It is shown that the *B* phase not only has a "phase dynamics" such as that in He II, but also an orbital dynamics somewhat similar to that of ^{3}He-*A* or a nematic liquid cyrstal. The rederived *B*-phase hydrodynamic theory, complete with nonlinear terms, establishes the equivalence between rotations in orbital and spin spaces and shows the intriguing possibilities of generating a magnetization by mechanical rotations and of substituting the magnetic field with oscillating parallel plates in "NMR" experiments.

The most exciting aspect of the two superfluid phases of ^{3}He is the richness of their spontaneously broken symmetries. The *B* phase, in particular, exhibits the rather subtle concept of spontaneously broken spin-orbit symmetry (SBSOS) introduced by Leggett, which is a broken symmetry only of *relative* spin-orbit rotations.[1] In previous studies of *B*-phase dynamics,[2,3] however, the consequences of SBSOS seem indistinguishable from those of the broken rotational symmetry in spin space alone, such as that present in an antiferromagnet or a spin glass. This is indeed disturbing because it defies the general belief in the intimate relation between the spontaneously broken symmetry and the hydrodynamics of a system. The discrepancy warrants the effort to reinvestigate *B*-phase dynamics. One can plausibly expect a system with SBSOS, in which the orbital variables are kept constant, to behave as if only the rotational symmetry in spin space were broken, and based on this argument, *B*-phase spin dynamics has been successfully investigated.[2] But we can also turn the argument around to conclude that keeping the spin variables constant, the system has to account for the three broken orbital symmetries (as would be present in the hypothetical biaxial nematics), and its dynamics must therefore differ from the two-fluid hydrodynamics of He II, where the only broken symmetry is the gauge invariance. So qualitatively speaking, *B*-phase dynamics will be given by the combination of the spin-glass dynamics of Halperin and Saslow[4] and superfluid biaxial nematodynamics with a number of elastic coefficients vanishing to reflect the symmetry of the Balian-Werthamer state, which is invariant un-

