Coherent States for General Potentials

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We define coherent states for general potentials, requiring that they have the physically interesting properties of the harmonic-oscillator coherent states. We exhibit these states for several solvable examples and show that they obey a quantum approximation to the classical motion.

The well-known[1-3] coherent states $|\alpha\rangle$ for the simple harmonic oscillator were originally obtained by Schrödinger[1] as those quantum states which obey the classical motion: $\langle\alpha|x(t)|\alpha\rangle = A\sin(\omega t+\varphi)$. These states have a number of other interesting properties including the following: (1) They minimize the uncertainty relation $(\Delta x)^2(\Delta p)^2 \geq \hbar^2/4$, and have $\Delta x = (\hbar/2m\omega)^{1/2}$. (2) They are eigenstates of the destruction operator: $a^-|\alpha\rangle = \alpha|\alpha\rangle$. (3) They are created from the ground state by a unitary displacement operator: $\{\exp[\alpha a^+ - \alpha^* a^-]\}|0\rangle = |\alpha\rangle$. These properties are all equivalent. In fact, usually one of them is adopted as the definition of the harmonic-oscillator coherent states.

For systems other than the harmonic oscillator, definitions of coherent states have been proposed based[4] on property (2) and[5] on property (3), but have been applied to systems having equal level spacing. In a general system the discrete energy levels are not equally spaced, there may be a continuous part of the spectrum, and the problem may not be solvable in closed form. In such cases these definitions are difficult to apply. Moreover, they represent a departure from the original motivation[1] for studying coherent states; namely, that they obey the classical motion. We seek a definition which retains this property and which is generally applicable.

Consider a one-dimensional, single-particle, quantum-mechanical system described by a local potential with one confining region.[6] Classically, the bound-state motion is periodic and by a suitable mapping one can find a function $X_c(x)$ which varies sinusoidally (as will its associated momentum $P_c = m\dot{X}_c$).

Specifically, if $m\ddot{x} = -dV(x)/dx$, then $X_c(x) \equiv A\sin(\omega_c t+\varphi)$ and $P_c = pX_c'(x)$ obey

$$m\dot{X}_c = P_c, \quad \dot{P}_c = -m\omega_c^2 X_c, \tag{1}$$

where

$$X_c'(x) \equiv \frac{dX_c}{dx} = \omega_c\left[\frac{\frac{1}{2}m(A^2 - X_c^2)}{E - V(x)}\right]^{1/2} \tag{2}$$

and

$$(2m)^{-1}P_c^2 + \tfrac{1}{2}m\,\omega_c^2 X_c^2 = \tfrac{1}{2}m\omega_c^2 A^2. \tag{3}$$

Note that, in general, ω_c and A depend upon the total energy E.

The corresponding quantum-mechanical operators are (to within overall normalizations which can be arranged for convenience)

$$X \equiv X_c, \quad P \equiv \frac{-i\hbar}{2}\left[\frac{d}{dx}X' + X'\frac{d}{dx}\right]. \tag{4}$$

They obey

$$[X, P] = i\hbar(X')^2 \tag{5}$$

and

$$(\Delta X)^2(\Delta P)^2/\langle(X')^2\rangle^2 \geq \tfrac{1}{4}\hbar^2. \tag{6}$$

The states which minimize this uncertainty relation satisfy[7]

$$A^-\psi_\alpha(x) \equiv \tfrac{1}{2}[X/\Delta X + iP/\Delta P]\psi_\alpha(x) = \alpha\psi_\alpha(x), \tag{7}$$

where $\alpha = \frac{1}{2}[\langle X\rangle/\Delta X + i\langle P\rangle/\Delta P]$. [For simplicity, we have assumed that X_c is independent of E. If not, then to obtain X, one must make the replacement $E \to H$, add a possible zero-point energy, and symmetrize. Such a more complicated $X_c(x, E)$ occurs for the Morse potential, which system will be discussed elsewhere.]

We claim that a subset of these minimum-uncertainty states, labeled by a particular value of $\Delta X/\Delta P$, are the appropriate generalization of the harmonic-oscillator coherent states. We shall now demonstrate that they obey the classical motion and suitably generalize other coherent-state properties.

Equations (4) imply that $\dot{X} = -i\hbar^{-1}[X, H] = P/m$, so that the first classical equation of motion (1) is obeyed. The second of the classical equations of motion (1) cannot be obeyed precisely because ψ_α is a superposition of energy eigenstates and in general ω_c depends upon E (the harmonic oscillator is exceptional in this regard). We shall demonstrate in our examples, however, that if $\dot{P}_c = -(K_1E + K_2)X_c$, then quantum mechanically $\dot{P}$

$\equiv -i\hbar^{-1}[P,H] = -\frac{1}{2}\{K_1 H + K_2 + Z, X\}$, where Z is a quantum correction and $\{,\}$ denotes the anticommutator. One obtains a quantum approximation to the classical motion, with correct amplitude and frequency, by specifying a particular value for $\Delta X/\Delta P$.

By construction the $\psi_\alpha(x)$ satisfy a generalized version of property (1). They are also [see Eq. (7)] eigenstates of the generalized annihilation operator A^- [property (2)]. For the required value of $\Delta X/\Delta P$, A^- will turn out to be the ground-state destruction operator, A_0^-. In general this relationship is not obvious, for the following reason. For a general potential, in contrast to the harmonic oscillator, the raising and lowering operators $A_n^\pm$ for the energy eigenstates E_n depend explicitly upon the state label n. Moreover, it is not generally true that $(A_n^-)^\dagger = A_n^+$. Nevertheless, it will turn out in our examples that the "natural" position and momentum variables are expressible in the form [$K(n)$ a c-number]

$$X = \tfrac{1}{4}K(n)\{[A_n^- + (A_n^+)^\dagger] + [A_n^+ + (A_n^-)^\dagger]\},$$
$$P = \frac{-i}{4}\{[A_n^- + (A_n^+)^\dagger] - [A_n^+ + (A_n^-)^\dagger]\}, \tag{8}$$

and these operators do not depend explicitly upon n. Equation (7) is therefore seen to be a generalization of property (2), with $a^- = (2m\hbar\omega)^{-1/2}[m\omega x + ip]$ replaced by A^-. The n dependence of $A_n^\pm$ makes the connection of our coherent states to property (3) more difficult to establish.

Note also, that in principle our definition of the coherent states can be implemented even for systems which cannot be solved in closed form. Equation (2) can be solved for X_c by either analytic or numerical approximation methods. Such a solution then suffices for the calculation of X and P via (4), and Eq. (7) can be solved for the coherent states $\psi_\alpha(x)$.

We now summarize the results of applying our method to several solvable examples.

(A) Harmonic oscillator.—For the case of the harmonic oscillator, the natural variables are the usual x and p, and our generalized coherent states reduce, as already stated, precisely to the familar coherent states,[1-3] with $\Delta x/\Delta p = 1/m\omega$.

(B) Symmetric Rosen-Morse potential.[8]—It is convenient to add U_0 to the usual $U_0 \cosh^{-2} ax$ and deal with $V(x) = U_0 \tanh^2 z$, $z \equiv ax$. Making the convenient choice $\varphi = 0$, the classical bound state solutions are $X_c = \sinh z = [E/(U_0 - E)]^{1/2} \sin\omega_c t$, where $\omega_c = [2a^2(U_0 - E)/m]^{1/2}$ and $U_0 > E$. For free particles, $U_0 < E$, the circular functions become hyperbolic functions. In either case, the equations of motion are $\dot{X}_c = P_c/m$, $\dot{P}_c = -2a^2(U_0 - E)X_c$.

The quantum operators $X = \sinh z$ and $P = -\frac{1}{2}i\hbar a^2 \times[\cosh z(d/dz) + (d/dz)\cosh z]$ obey $[X, P] = i\hbar a^2 \times\cosh^2 z$, and the normalized minimum-uncertainty states are

$$\psi = \left[\frac{a\,\Gamma(B+\frac{1}{2}+iu)\,\Gamma(B+\frac{1}{2}-iu)}{\pi^{1/2}\,\Gamma(B)\,\Gamma(B+\frac{1}{2})}\right]^{1/2} (\cosh z)^{-B} \exp[C \sin^{-1}(\tanh z)], \tag{9}$$

where $B = \frac{1}{2}[\langle\cosh^2 z\rangle/(\Delta\sinh z)^2 + 1]$ and $C = u + iv = B\langle\sinh z\rangle + \langle\cosh z(d/dz)\rangle$. One can verify that (9) satisfies (7) and that $(\Delta X)^2(\Delta P)^2 = \frac{1}{4}\hbar^2 a^4 \langle\cosh^2 z\rangle^2$. The second equation of motion is $\dot{P} = -a\{U_0 - H - \frac{1}{4}E_0, X\}$, where $E_0 = a^2\hbar^2/2m$. These allow one to calculate $X(t)$ exactly. A quantum approximation to the classical motion follows if $B = s$, where $U_0 = E_0 s(s+1)$.

The normalized energy eigenfunctions[9] are $[a(s-n)\,\Gamma(2s-n+1)/\Gamma(n+1)]^{1/2} P_s^{(n-s)}(\tanh z)$. It follows that $A_n^\pm = (s-n)\sinh z \mp \cosh z(d/dz)$. Therefore, Eq. (8) yields X and P in agreement with the forms obtained from X_c and P_c.

It is interesting to note that in the limit $a \to 0$, $a^2 s \to m\omega/\hbar$, the potential, eigenfunctions, eigenvalues, X, P, and the coherent states as defined here all approach their counterparts for the harmonic oscillator.

(C) Symmetric Pöschl-Teller potential.[10]—Here it is convenient to subtract $U_0 \equiv \lambda(\lambda-1)E_0$ from the usual form $U_0 \cos^{-2} z$ and deal with $V(x) = U_0 \tan^2 z$, where $z = ax$. The problem is mathematically similar to that for the Rosen-Morse potential (except that there is no continuum). The "natural" quantum operators are $X = \sin z$ and $P = -\frac{1}{2}i\hbar a^2[\cos z(d/dz) + (d/dz)\cos z]$, which obey $[X, P] = i\hbar a^2 \cos^2 z$. The normalized minimum-uncertainty states are found to be

$$\psi = \left[\frac{a}{\pi^{1/2}}\,\frac{\Gamma(B+\frac{1}{2})\,\Gamma(B+1)}{\Gamma(B+\frac{1}{2}+u)\,\Gamma(B+\frac{1}{2}-u)}\right]^{1/2} (\cos z)^{B} \left[\frac{1+\sin z}{1-\sin z}\right]^{C/2}, \tag{10}$$

where $B=\frac{1}{2}[-1+\langle\cos^2 z\rangle/(\Delta \sin z)^2]$, $C=u+iv=\langle\cos z(d/dz)\rangle+B\langle\sin z\rangle$. Again, except for quantum corrections, the second classical equation of motion is obtained: $\dot{P}=-a^2\{U_0+H-\frac{1}{4}E_0, X\}$, and one can claculate $X(t)$ exactly. The classical motion follows when $B=\lambda$.

The normalized energy eigenfunctions[9] can be expressed in terms of Legendre functions and the raising and lowering operators are analogous to those for case B. All of these results approach those for the harmonic oscillator in the limit $a\to 0$, $\lambda a^2 \to m\omega/\hbar$.

(D) Infinite square well.—In the limit $\lambda\to 1$ the symmetric Pöschl-Teller potential approaches the infinite square well, with walls at $\pm d=\pm\pi/2a$. Notice, however, that a minimum-uncertainty state for the potential $V(x)=U_0 u(ax)$ is also[11] a minimum-uncertainty state for the potential $U_0'\times u(ax)$. It follows that any minimum-uncertainty state for a symmetric Pöschl-Teller potential is also a minimum-uncertainty state for the corresponding infinite square well (with $d=\pi/2a$). In particular, even though any state in a flat well has difficulty producing the classical motion, Eq. (10), with $B=1$, is a coherent state for the infinite square well.

Observe that the first classical equation of motion for the Pöschl-Teller potential becomes $\sin(\pi x/2d)=\sin\omega t$. This is indeed a correct expression for the behavior of a classical particle confined by rigid walls.

Finally, we briefly consider problems in three dimensions. As in the one-dimensional case, the coherent states should reproduce, as closely as possible, the classical motion. We therefore use the classical problem as a guide. A problem which is separable, such as the three-dimensional harmonic oscillator, can be treated trivally by analogy to the one-dimensional case. More generally, one must deal with non-Cartesian coordinate systems.

In the following example, we limit ourselves to a spherically symmetric potential and treat the radial portion of the problem.[12] One must realize that the natural angular variable is no longer the time but a generalized variable $\varphi(t)$ which varies between successive apsidal distances: $\dot{\varphi}(t)=L/mr^2(t)$.

(E) Coulomb potential.—The classical Kepler solution is $X_c=(1/r-me^2/L^2)=A\sin\varphi(t)$, where $A=[(m^2e^4/L^4)+(2mE/L^2)]^{1/2}$. $P_c=(p_r)_c=-AL\times\cos\varphi(t)$ and one has $[me^4/(2L^2)+E]=\frac{1}{2}P_c{}^2/m+\frac{1}{2}L^2X_c{}^2/m$.

The quantum operators are $X=X_c$, with $L^2=\hbar^2 l(l+1)$, and $P=p_r\equiv -i\hbar[(d/dr)+1/r]$. The resulting minimum-uncertainty states are

$$\psi=(2u)^{B+1/2}[\Gamma(2B+1)]^{-1/2}r^{B-1}e^{-Cr}, \qquad (11)$$

with $B=\frac{1}{2}\langle 1/r^2\rangle/[\Delta(1/r)]^2$, and $C=u+iv=B\langle 1/r\rangle-i\langle P\rangle/\hbar$. One finds also that X and P are related as in Eq. (8) to the operators $A_l^{\pm}$ which raise and lower l for the radial eigenfunctions $R_{nl}(r)$. Here the "ground-state" annihilation operator is $A_{n-1}{}^{+}$, which indicates that $B=n$ for the coherent states.[13]

Further details and other results, including numerical studies of the time evolution of our coherent states, will appear elsewhere.

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Tensor Mesons as a Source of Low-Mass Dimuons

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Recently, anomalous production of dimuons with $m < 600$ MeV has been reported in 16-GeV/c $\pi^\pm p$ collisions. Production and subsequent decay of the tensor mesons f and A_2^0 is suggested as a source for these dimuons.

Dilepton production in hadronic collisions is currently of considerable interest. At invariant masses greater than 1 GeV there are narrow resonances superimposed on a steeply falling continuum. The continuum is understood in terms of the parton-antiparton annihilation process proposed by Drell and Yan[1] and the resonances are taken as evidence of new quark flavors. However, a recent investigation[2] of low-mass dimuons produced in $\pi^\pm p$ collisions at 16 GeV/c has found dimuon production which does not easily fit into this scheme. The ρ and ω resonances are clearly seen, but below the ρ mass there remain contributions in addition to the known Dalitz decays.

In order to show what signal remains to be explained, known contributions are subtracted from the histogram of Bunnell *et al.*[2] in the following way: (i) Their expected (rather than their maximum) Dalitz-decay signal is subtracted, and (ii) except for two events per bin all events between 0.62 and 0.89 GeV are assigned to the vector mesons. Two events per bin is the average number found in bins immediately to either side of the resonances. Fourteen events in the bin centered on 0.785 GeV are assigned to ω, this number being chosen to give a smooth ρ peak. The rest of the resonance events are assigned to ρ.

The original data suggest that the ρ and ω mesons are superimposed upon a smoothly falling background. However, when the Dalitz contribution, which is surely present even if not in precisely the amounts I have assumed, is subtracted, what remains invites the following interpretation: There is a flat continuum from threshold to a sharp cutoff around 0.55 GeV, superimposed on a slowly varying background of about two events per bin. The statistics are such that the 0.55-GeV dip may be a fluctuation, but the data are at least consistent with the interpretation suggested and I wish to propose a mechanism which accounts for this shape. It is difficult to obtain the required smoothly falling spectrum. In particular, simple quark counting arguments imply for a Drell-Yan mechanism

$$\frac{\pi^- p \to \mu^+\mu^- X}{\pi^+ p \to \mu^+\mu^- X} \sim 8$$

(see for example Donnachie and Landshoff[3]) whereas the observed ratio[2] is 1.28 ± 0.23.

Given a flat distribution with a sharp cutoff, what is required is a decay of the form $h \to h'\mu^+\mu^-$, where h and h' are hadrons with

$$\Delta M = M_h - M_{h'} \approx 0.55 \text{ GeV}. \qquad (1)$$

The Dalitz decays of η and ω have this feature, but being p-wave decays their spectra near the upper threshold (where the dimuon invariant mass is $m \lesssim \Delta M$) are proportional to $(\Delta M - m)^{3/2}$. A sharp cutoff requires an s-wave decay (electric dipole) with $(\Delta M - m)^{1/2}$ threshold behavior. I propose that the most important candidates are

$$\text{I: } A_2^0 \to \omega\mu^+\mu^-, \quad \Delta M_\text{I} = 0.53 \text{ GeV},$$

$$\text{II: } f \to \rho^0\mu^+\mu^-, \quad \Delta M_\text{II} = 0.50 \text{ GeV}.$$

This proposal is tested in two stages, by first checking that the spectra obtained have the right shape, particularly in respect of the position of

