PHYSICAL REVIEW LETTERS

VOLUME 41 | 24 JULY 1978 | NUMBER 4

Construction of Solutions of Gravitational, Electromagnetic, or Other Perturbation Equations from Solutions of Decoupled Equations

Robert M. Wald
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
(Received 15 May 1978)

A simple proof is given of a general result which shows how to construct solutions of a coupled system of linear self-adjoint partial differential equations once one has succeeded in deriving a decoupled equation in the manner described below. In the case of electromagnetic and gravitational perturbations of algebraically special vacuum space-times, this procedure yields the formulas of Cohen and Kegeles and of Chrzanowski.

In general relativity, as in most other branches of physics, one is often faced with the task of solving a coupled system of linear partial differential equations. The equations describing the propagation of small electromagnetic or gravitational disturbances in a space-time are prime examples of such systems. Much effort has been devoted to solving these equations, but except in the simplest cases (such as Robertson-Walker or Schwarzschild space-times) direct attacks on the full system of equations generally have not been successful.

In several cases, however, it has been possible to derive from the original system a decoupled equation for a new variable. The most important example of this is the case of algebraically special vacuum space-times, where, by the method of Teukolsky,[1] one can derive from the electromagnetic perturbation equations a decoupled equation for the Newman-Penrose[2] component φ_0 of the Maxwell field, and from the gravitational perturbation equations, one can obtain a decoupled equation for the Newman-Penrose component ψ_0 of the perturbed Weyl tensor. In the case of a Kerr black hole (which is a type-D solution, so that additional decoupled equations for φ_2 and ψ_4 can also be derived), these equations can be solved by separation of variables. Furthermore, many physically interesting quantities (such as the ingoing or outgoing radiation fluxes) can be calculated directly from the decoupled quantities, and so the solution of many physical problems can be obtained by this means. However, for some problems, one needs to know the complete electromagnetic or gravitational perturbation. For the Kerr metric, Chandrasekhar[3] recently has succeeded in systematically solving the complete gravitational perturbation equations, but the problem remains for other space-times.

Progress toward obtaining the complete perturbations for vacuum algebraically special space-times has been made by Cohen and Kegeles[4,5] and Chrzanowski.[6] In the electromagnetic case, Cohen and Kegeles[4] obtained an equation for a potential ψ_E from which solutions of the full Maxwell equations can be generated by differentiation. Making the conjecture that the Green's function for the full electromagnetic and gravitational perturbations of Kerr could be expressed in a certain factorized form, Chrzanowski[6] derived formulas for the vector-potential and metric perturbations of Kerr. In the electromagnetic case his construction is equivalent to that of Cohen and Kegeles.[4] On the basis of this analogy, Chrzanowski conjectured a formula—subsequently also given (without derivation) by Cohen and Kegeles[5]—for the metric perturbation of an arbitrary vacuum algebraically special space-time in terms of the solutions of an equation for a potential ψ_G. Both the Cohen and Kegeles derivation for electromagnetic perturbations and the Chrzanowski derivation for Kerr are rather complicated and rely on specific properties of (and, in Chrzanowski's case, unproven assumptions about) the particular system of equations under consideration.

In this Letter, I shall give a remarkably simple proof of a completely general result which states that, whenever a decoupled equation can be derived from a system of linear, partial differential equations in the manner specified below, then a solution of the adjoint equation to the decoupled equation generates (by direct differentiation) a solution of the system of equations adjoint to the original system (and thus a solution of the original system in the self-adjoint case). For electromagnetic and gravitational perturbations of vacuum, algebraically special space-times, this result yields the formulas of Cohen and Kegeles and Chrzanowski.

Let M be a smooth (C^∞) manifold with smooth metric $g_{\mu\nu}$ and derivative operator ∇_μ associated with the metric. The presence of the metric is purely for convenience; with a suitable redefinition of adjoint the theorem proven below remains valid for an arbitrary derivative operator. Similarly, the smoothness assumptions can be weakened. I wish to consider linear partial differential operators on M, mapping n-index smooth tensor fields into m-index tensor fields. By a "linear partial differential operator" I mean precisely an operator which can be expressed as a finite sum of smooth tensor fields contracted with the derivative operator ∇_μ. I shall denote such operators by script capital letters. Again, the results below can easily be generalized to encompass operators which map collections of tensor fields into collections of tensor fields (as one would need to consider, for example, when treating the coupled Einstein-Maxwell perturbations of electrovac space-times with a nonzero background electromagnetic field).

Suppose we wish to solve the equation

$$\mathcal{E}(f)=0, \tag{1}$$

where $\mathcal{E}$ is a linear partial differential operator and f is a tensor field of the type on which $\mathcal{E}$ acts. Prime examples of $\mathcal{E}$ in general relativity are the Maxwell operator describing electromagnetic perturbations,

$$[\mathcal{E}_E(A_\lambda)]_\mu=\nabla^\nu\nabla_\nu A_\mu-\nabla^\nu\nabla_\mu A_\nu, \tag{2}$$

and the linearized Einstein operator describing gravitational perturbations of vacuum space-times,

$$[\mathcal{E}_G(h_{\rho\sigma})]_{\mu\nu}=-\nabla_\mu\nabla_\nu h^\alpha{}_\alpha-\nabla^\alpha\nabla_\alpha h_{\mu\nu}+\nabla^\alpha\nabla_\nu h_{\alpha\mu}+\nabla^\alpha\nabla_\mu h_{\alpha\nu}+g_{\mu\nu}(\nabla^\alpha\nabla_\alpha h^\beta{}_\beta-\nabla^\alpha\nabla^\beta h_{\alpha\beta}). \tag{3}$$

Suppose a decoupled equation has been derived in the following manner: A new variable $\varphi=\mathcal{T}(f)$ (where $\mathcal{T}$ is a linear partial differential operator) has been introduced and a linear partial differential operator $\mathcal{S}$ has been found such that for all f,

$$\mathcal{S}\mathcal{E}(f)=\mathcal{O}\mathcal{T}(f)=\mathcal{O}(\varphi), \tag{4}$$

where $\mathcal{O}$ is yet another linear partial differential operator. The equation $\mathcal{E}(f)=0$ then implies the "decoupled equation" $\mathcal{O}(\varphi)=0$.

To make the above statements more concrete, consider the example mentioned above of electromagnetic perturbations of vacuum, algebraically special space-times. The new variable φ is chosen to be the Newman-Penrose component φ_0 of the Maxwell field and the operator $\mathcal{T}_E$ is the formula for φ_0 in terms of A_λ,

$$\mathcal{T}_E(A_\lambda)=l^\mu m^\nu(\nabla_\mu A_\nu-\nabla_\nu A_\mu), \tag{5}$$

where, following standard conventions, l^μ and m^μ are Newman-Penrose tetrad vectors with l^μ aligned along the repeated principal null direction. The operator $\mathcal{S}_E$ describes the manipulations that must be performed on the Maxwell equations to derive the Teukolsky equation. It can be read off from the source term of the inhomogeneous Teukolsky equation[1,6] and, in Newman-Penrose notation, is given by

$$2\mathcal{S}_E(J_\lambda)=(\delta-\beta-\bar{\alpha}-2\tau+\bar{\pi})(J_\mu l^\mu)-(D-\epsilon+\bar{\epsilon}-2\rho-\bar{\rho})(J_\mu m^\mu). \tag{6}$$

Teukolsky's derivation[1] shows that the operator identity

$$\mathcal{S}_E\mathcal{E}_E=\mathcal{O}_E\mathcal{T}_E \tag{7}$$

holds, where $\mathcal{O}_E$ is the Teukolsky operator, defined by

$$\mathcal{O}_E(\varphi)=(D-\epsilon+\bar{\epsilon}-2\rho-\bar{\rho})(\Delta+\mu-2\gamma)\varphi - (\delta-\beta-\bar{\alpha}-2\tau+\bar{\pi})(\bar{\delta}+\pi-2\alpha)\varphi. \quad (8)$$

Exactly similar relations hold for gravitational perturbations.[1]

Before stating the main result, I need to introduce the notion of adjoints. As most readers undoubtedly know, for a linear partial differential operator $\mathcal{L}$ taking scalar fields into scalar fields, it is always possible (because of the Leibnitz property of the derivative operator ∇_μ) to express uniquely the product $\psi(\mathcal{L}\varphi)$, where φ and ψ are scalar fields, as the product of φ with a derivative operator $\mathcal{L}^\dagger$ acting on ψ, plus a total divergence:

$$\psi(\mathcal{L}\varphi)-(\mathcal{L}^\dagger\psi)\varphi=\nabla_\mu s^\mu. \quad (9)$$

$\mathcal{L}^\dagger$ is called the adjoint of $\mathcal{L}$. The generalization of this notion to an operator $\mathcal{P}$ taking n-index tensor fields to m-index tensor fields is straightforward. The adjoint, $\mathcal{P}^\dagger$, of $\mathcal{P}$ takes m-index tensor fields into n-index tensor fields and is defined by

$$\psi^{\mu_1\cdots\mu_m}(\mathcal{P}\varphi)_{\mu_1\cdots\mu_m}-(\mathcal{P}^\dagger\psi)^{\nu_1\cdots\nu_n}\varphi_{\nu_1\cdots\nu_n}=\nabla_\mu t^\mu. \quad (10)$$

As with scalar operators, the composition of tenfor operators $\mathcal{P}$ and $\mathcal{Q}$ satisfied $(\mathcal{P}\mathcal{Q})^\dagger=\mathcal{Q}^\dagger\mathcal{P}^\dagger$. We say $\mathcal{P}$ is self-adjoint if $\mathcal{P}^\dagger=\mathcal{P}$ (which is possible, of course, only if $m=n$). It is easy to check that the operators $\mathcal{E}_E$ and $\mathcal{E}_G$ given by Eqs. (2) and (3) are self-adjoint.

I now shall prove the following result:

Theorem.—Suppose the identity $\mathcal{S}\mathcal{E}=\mathcal{O}\mathcal{T}$ holds for the linear partial differential operators $\mathcal{S}$, $\mathcal{E}$, $\mathcal{O}$, and $\mathcal{T}$. Suppose ψ satisfies $\mathcal{O}^\dagger\psi=0$. Then $\mathcal{S}^\dagger\psi$ satisfies $\mathcal{E}^\dagger(\mathcal{S}^\dagger\psi)=0$. Thus, in particular, if $\mathcal{E}$ is self-adjoint then $\mathcal{S}^\dagger\psi$ is a solution of $\mathcal{E}(f)=0$.

Proof.—Taking the adjoint of $\mathcal{S}\mathcal{E}=\mathcal{O}\mathcal{T}$ we have

$$\mathcal{E}^\dagger\mathcal{S}^\dagger=\mathcal{T}^\dagger\mathcal{O}^\dagger. \quad (11)$$

Applying these operators to ψ, we obtain

$$\mathcal{E}^\dagger\mathcal{S}^\dagger\psi=0, \quad (12)$$

which is the desired result.

As an application of this result, consider electromagnetic perturbations of vacuum, algebraically special space-times. The adjoint of $\mathcal{O}_E$ is easily computed to be

$$\mathcal{O}_E^\dagger=(\Delta+\bar{\mu}+\gamma-\bar{\gamma})(D+2\epsilon+\rho) - (\bar{\delta}+\alpha+\bar{\beta}-\bar{\tau})(\delta+2\beta+\tau). \quad (13)$$

The equation $\mathcal{O}_E^\dagger\psi_E=0$ is then recognized as precisely the Cohen-Kegeles equation (4). The above theorem proves that if ψ_E is a solution of the Cohen-Kegeles equation, then

$$A^\mu=2[\mathcal{S}_E^\dagger(\psi_E)]^\mu = -l^\mu(\delta+2\beta+\tau)\psi_E+m^\mu(D+2\epsilon+\rho)\psi_E, \quad (14)$$

is a vector-potential solution of Maxwell's equation. Equation (14) is precisely the complex conjugate of Chrzanowski's[6] Eq. (6.7). [Real solutions are obtained by taking the real and imaginary parts of (14), and so (14) is, of course, completely equivalent to Chrzanowski's result.] Differentiation of the real part of Eq. (14) to obtain the Maxwell field tensor components yields the Cohen and Kegeles[4] formulas. [The imaginary part of Eq. (14) yields the duality-rotated solution.]

In an entirely similar manner, for gravitational perturbations of vacuum, algebraically special space-times the adjoint equation $\mathcal{O}_G^\dagger\psi_G=0$ for the Teukolsky operator $\mathcal{O}_G$ is just Chrzanowski's Eq. (6.11) which is the same as Eq. (5) of Cohen and Kegeles.[5] (Note that Chrzanowski's rather unnatural procedure for obtaining this equation is equivalent to simply taking the adjoint.) Application of $\mathcal{S}_G^\dagger$ yields the complex conjugate of Chrzanowski's[6] Eq. (6.13) [Cohen-Kegeles's[5] Eq. (6)] for the metric perturbation in terms of ψ_G. [Note, however, that the Cohen-Kegeles formulas for the Newman-Penrose Weyl tensor components are somewhat misleading since they do not take the real part of the metric perturbation before calculating these components. Hence, these components do not necessarily arise from a real metric perturbation. Formulas for ψ_0 and ψ_4 for real metric perturbations in the type-*D* case are given below in Eqs. (17) and (18).] Thus, the results of Cohen and Kegeles and Chrzanowski can be derived in a very simple manner as special cases of the above theorem. Further applications will be given elsewhere.

It is interesting to note that if ψ satisfies $\mathcal{O}^\dagger\psi=0$ then, as proven above, $\mathcal{E}^\dagger\mathcal{S}^\dagger\psi=0$, and hence, if $\mathcal{E}$ is self-adjoint, $0=\mathcal{S}\mathcal{E}\mathcal{S}^\dagger\psi=\mathcal{O}(\mathcal{T}\mathcal{S}^\dagger\psi)$. Thus, if $\mathcal{E}$ is self-adjoint the operator $\mathcal{T}\mathcal{S}^\dagger$ maps solutions of the adjoint equation $\mathcal{O}^\dagger\psi=0$ into solutions of the equation $\mathcal{O}\varphi=0$. In the case of electromagnetic perturbations of vacuum type-*D* space-times, it turns out that the Teukolsky equation for φ_2 implies that the quantity $(\psi_2{}^0)^{-2/3}\varphi_2$ satisfies the adjoint of the Teukolsky equation for φ_0, where

$\psi_2{}^0$ is the Weyl tensor component ψ_2 of the unperturbed space-time. Thus, if φ_E is a solution of Teukolsky's equation for $(\psi_2{}^0)^{-2/3}\varphi_2$, then the quantity

$$\varphi_0 = 4\mathcal{T}_E(\mathrm{Re}\,\mathcal{S}_E{}^\dagger\varphi_E) = -(D+\bar{\epsilon}-\epsilon-\bar{\rho})(D+2\bar{\epsilon}+\bar{\rho})\bar{\varphi}_E \tag{15}$$

is a solution of Teukolsky's equation for φ_0, where $\mathcal{T}_E$ and $\mathcal{S}_E{}^\dagger$ are given by Eqs. (5) and (14) above, and Re denotes the operation of taking the real part. The field tensor component φ_2 associated with the real vector potential solution generating φ_0 via Eq. (15) is *not* the original solution $(\psi_2{}^0)^{2/3}\varphi_E$, but rather

$$\varphi_2 = 4\mathcal{T}_E{}'(\mathrm{Re}\,\mathcal{S}_E{}^\dagger\varphi_E) = -(\bar{\delta}-\bar{\tau}+\alpha+\bar{\beta})(\bar{\delta}+2\bar{\beta}+\bar{\tau})\bar{\varphi}_E. \tag{16}$$

In an exactly similar manner, for gravitational perturbations of vacuum type-D space-times, ψ_0 and $(\psi_2{}^0)^{-4/3}\psi_4$ satisfy adjoint equations. Here we find that if ψ_G is a solution of the Teukolsky equation for $(\psi_2{}^0)^{-4/3}\psi_4$ then

$$\psi_0 = 4\mathcal{T}_G(\mathrm{Re}\,\mathcal{S}_G{}^\dagger\psi_G) = (D-\bar{\rho}-3\epsilon+\bar{\epsilon})(D-\bar{\rho}-2\epsilon+2\bar{\epsilon})(D-\bar{\rho}+3\bar{\epsilon}-\epsilon)(D+3\bar{\rho}+4\bar{\epsilon})\bar{\psi}_G \tag{17}$$

is a solution of the Teukolsky equation for ψ_0 where Eqs. (6.13) and (B11) of Ref. 6 have been used. The solution for ψ_4 associated with this solution for ψ_0 is

$$\begin{aligned}\psi_4 = {} & (\bar{\delta}-\bar{\tau}+3\alpha+\bar{\beta})(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})(\bar{\delta}+\alpha+3\bar{\beta}-\bar{\tau})(\bar{\delta}+4\bar{\beta}+3\bar{\tau})\bar{\psi}_G \\ & +\{(\bar{\delta}-\bar{\tau}+3\alpha+\bar{\beta})(\bar{\delta}-\bar{\tau}+2\alpha+2\bar{\beta})(\delta+\bar{\alpha}+3\beta-\tau)(\delta+4\beta+3\tau) \\ & +(\Delta+\bar{\mu}+3\gamma-\bar{\gamma})(\Delta+\bar{\mu}+2\gamma-2\bar{\gamma})(D-\rho+3\epsilon-\bar{\epsilon})(D+3\rho+4\epsilon) \\ & -[(\Delta+\bar{\mu}+3\gamma-\bar{\gamma})(\bar{\delta}-2\bar{\tau}+2\alpha)+(\bar{\delta}-\bar{\tau}+3\alpha+\bar{\beta})(\Delta+2\bar{\mu}+2\gamma)] \\ & \quad\times[(D+\bar{\rho}-\rho+\bar{\epsilon}+3\epsilon)(\delta+4\beta+3\tau)+(\delta+3\beta-\bar{\alpha}-\bar{\pi}-\tau)(D+3\rho+4\epsilon)]\}\psi_G.\end{aligned} \tag{18}$$

Equations (15)–(18) are the electromagnetic and gravitational Starobinsky-Teukolsky relations[3,7] for perturbations of an arbitrary vacuum type-D space-time. For the Kerr metric they may be simplified (by manipulations along the lines of Appendix C of Ref. 6) to yield relations between the radial and angular solutions for φ_0 and φ_2 and for ψ_0 and ψ_4 although I have not completed the reduction of Eq. (18), which evidently requires considerable algebra.

Finally, it is worth noting that Chrzanowski derived his results by postulating a factorized form of the Green's function for electromagnetic and gravitational perturbations of Kerr. Now that his final results are rigorously established, one may reverse the steps of Chrzanowski's argument (taking into account the known factorized form of the Green's function for the Teukolsky equation) to prove that his assumed factorized form of the Green's function for radiative modes of the complete electromagnetic and gravitational perturbations is indeed valid.

I wish to thank R. Geroch for a helpful comment and S. Chandrasekhar for sustaining my interest in these problems through numerous discussions during the past three years.

This research was supported in part by the National Science Foundation Grant No. PHY76-81102, and by the Alfred P. Sloan Foundation.

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