Phases of Two-Dimensional Heisenberg Spin Systems from Strong-Coupling Expansions

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A quantum-mechanical Hamiltonian formulation of lattice spin systems is used to search for phase transitions in the O(2), O(3), and O(4) Heisenberg models in two dimensions. Strong-coupling expansions which have been calculated through eighth order indicate a phase transition at nonzero coupling for the O(2) model, but the non-Abelian models are predicted to exist only in the strong-coupling phase.

We have studied the O(2), O(3), and O(4) Heisenberg models in two dimensions using quantum-mechanical Hamiltonian strong-coupling methods.[1] Series expansions for these theories' mass gaps *and* β functions yield the following results: For the *O(2) model* there is a phase transition at nonzero coupling where the mass gap vanishes with an (apparent) essential singularity. For the *O(3)* and *O(4) models* there are *no* phase transitions at nonzero coupling g; interpolation formulas for these theories' β functions indicate that the transition from weak to strong coupling occurs over a narrow region in g.

These results are interesting for several reasons. First, high-temperature expansions using the Euclidean formulation of these theories on two-dimensional square lattices have not yielded decisive results.[2] The method presented here, however, clearly separates the Abelian from the non-Abelian models. Second, Migdal[3] and Polyakov[4] have suggested that the phase diagrams of these two-dimensional spin systems are similar to those of various lattice gauge theories in four dimensions. Third, the Euclidean formulation of the O(3) model has instantons[5] while the O(4) model does not, and so it is interesting to search for qualitative differences between the two models' mass gaps, etc.

Our calculations proceed as follows. Consider a two-dimensional square lattice and place n-dimensional unit vectors $\vec{n}$ at each site. Nearest-neighbor spins are coupled through their inner product, so that the system's Hamiltonian is

$$\beta H = -K \sum_{\vec{m},\vec{u}} \vec{n}(\vec{m})\cdot\vec{n}(\vec{m}+\vec{u}), \tag{1}$$

where $m=(m_1,m_2)$ labels the lattice sites, the set $\{\vec{u}\}$ consists of the two independent unit lattice vectors, and $K=J/kT$. Consider the transfer matrix which "propagates" the system of spins in the y_1 direction. If one takes the continuum limit in the y_1 direction and relabels that axis "time," $y_1 \to it$, a quantum Hamiltonian description of the theory results.[6] Its derivation follows familiar steps, and so we simply quote the resulting quantum-mechanical Hamiltonian,

$$H=(g/2a)\textstyle\sum_m[\vec{J}^2(m) - x\vec{n}(m)\cdot\vec{n}(m+1)], \tag{2}$$

where a is the spatial lattice spacing, m is an integer which labels the sites, $\vec{J}$ is the angular momentum operator, g is a coupling constant related to K, and $x=2/g^2$. Another example of the passage from Eq. (1) to Eq. (2) is afforded by the Ising model. Then the equivalent of Eq. (2) is

$$H=(g/2a)\textstyle\sum_m[1-\sigma_3(m) - x\sigma_1(m)\sigma_1(m+1)], \tag{3}$$

where the $\sigma_i(m)$ are Pauli matrices. Equation (3) is solvable[7] and its critical indices are those of the Ising model formulated on a square lattice. This last fact is an example of universality; i.e., the critical behavior of the system is independent of its detailed lattice structure. It is not known if analogous universality statements hold for the O(n) spin systems. However, since our main concern is just the existence of phase transitions, we can ignore such delicate issues. We shall, however, check the series expansion for the O(2) model's mass gap against a functional form obtained from the square-lattice formulation of the model and find evidence for agreement.

To develop strong-coupling expansions one defines the dimensionless operator

$$W \equiv (2a/g)H = W_0 - xV, \quad (4a)$$

where

$$W_0 = \sum_m \vec{J}^2(m), \quad V = \sum_m \vec{n}(m)\cdot\vec{n}(m+1). \quad (4b)$$

The eigenvalues of W can be calculated in perturbation theory around those of W_0. For example, in the O(3) model the zero-momentum state of a spin wave is

$$|j\rangle = (3/N)^{1/2}\sum_m n_j(m)|0\rangle, \quad (5)$$

where j labels its polarization, N is the number of links of the lattice, and $|0\rangle$ is the strong-coupling vacuum $[\vec{J}^2(m)|0\rangle = 0]$. The theory's mass gap is then

$$m = (g/2a)F(x) = (g/2a)(2 - \tfrac{2}{3}x + \ldots), \quad (6)$$

where the first two terms (which the reader can easily check) in the expansion for F have been recorded.

A computer has produced the following results for the various models: O(2),

$$F(x) = 1 - x - \tfrac{1}{8}x^2 + 0.031x^3 + (1.44\times10^{-2})x^4 + (6.00\times10^{-3})x^5 + (2.26\times10^{-4})x^6 + (6.96\times10^{-4})x^7 - (1.75\times10^{-5})x^8 + \ldots; \quad (7a)$$

O(3),

$$F = 2 - \tfrac{2}{3}x + (3.70\times10^{-2})x^2 + (4.32\times10^{-3})x^3 + (3.27\times10^{-4})x^4 + (2.01\times10^{-5})x^5 - (1.69\times10^{-5})x^6 + \ldots; \quad (7b)$$

O(4),

$$F = 3 - \tfrac{1}{4}x + 0.0156x^2 + (1.14\times10^{-3})x^3 + (1.25\times10^{-5})x^4 - (1.08\times10^{-6})x^5 - (7.57\times10^{-7})x^6 + \ldots. \quad (7c)$$

For the Ising model the strong-coupling expansion truncates after first order and one finds $F = 2 - 2x$, exactly. So, the mass gap vanishes at $x = 1$ with the critical index $\nu = 1$. These are exact results.[7]

A sound method of searching for phase transitions in general is to look for zeros in each theory's β function,

$$\beta(g) \equiv a\,dg/da. \quad (8)$$

Its strong-coupling expansion follows from the requirement that the theory's mass gap be a physical quantity independent of a:

$$dm/da = 0. \quad (9)$$

Using Eq. (6), we find

$$\beta(g)/g = F(x)/[F(x) - 2xF'(x)]. \quad (10)$$

Note that a vanishing mass gap produces a zero in $\beta(g)$; i.e., a vanishing gap is a signal for a second- (or higher-) order phase transition.

The strong-coupling series for $\beta(g)/g$ are as follows: O(2),

$$\beta(g)/g = 1 - 2x + 2.5x^2 - 3.06x^3 + 3.81x^4 - 4.71x^5 + 5.80x^6 - 7.13x^7 + 8.78x^8 + \ldots; \quad (11a)$$

O(3),

$$\beta(g)/g = 1 - \tfrac{2}{3}x + 0.296x^2 - 0.123x^3 + (5.15\times10^{-2})x^4 - (2.15\times10^{-2})x^5 + (8.87\times10^{-3})x^6 + \ldots; \quad (11b)$$

O(4),

$$\beta(g)/g = 1 - \tfrac{1}{3}x + (7.64\times10^{-2})x^2 - (1.57\times10^{-2})x^3 + (3.20\times10^{-3})x^4 - (6.47\times10^{-4})x^5 + (1.28\times10^{-4})x^6 + \ldots. \quad (11c)$$

We have searched for phase transitions by studying the series for F and β using several methods. Consider the ratio test. The general idea behind it is to suppose that a function has a power-law singular-

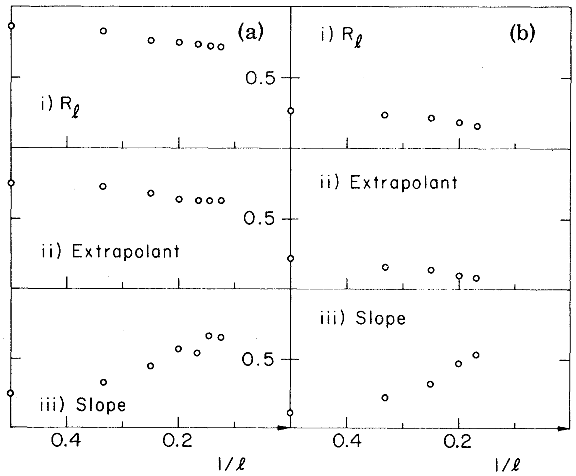

FIG. 1. Ratios, extrapolants, and slopes (defined in the text) vs $1/l$ for (a) the O(2) and (b) the O(3) models.

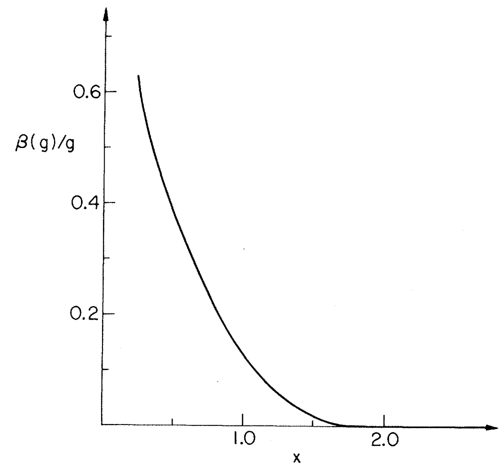

FIG. 2. $\beta(g)/g$ vs $x=2/g^2$ for the O(2) model.

ity at x_c:

$$f(x) \underset{x \to x_c}{\simeq} b(x-x_c)^{-\rho}. \tag{12}$$

Then the ratio of coefficients of the power series $f=\sum_l a_l x^l$ should obey the law

$$R_l \equiv a_l/a_{l-1} \underset{l\to\infty}{\simeq} x_c^{-1}[1+(\rho-1)/l]. \tag{13}$$

Consider these quantities for the reciprocal of the mass gap, $[F(x)]^{-1}$. In Figs. 1(a) and 1(b) we show for the O(2) and O(3) models the following: (i) R_l vs $1/l$; (ii) the linear extrapolant $lR_l-(l-1)R_{l-1}$ vs $1/l$, which gives an estimate of x_c^{-1}; (iii) the slope $(R_l-R_{l-1})[1/l-1/(l-1)]^{-1}$, which gives an estimate of $x_c^{-1}(\rho-1)$. The results are as follows: For the *O(2) model* the ratios of $[F(x)]^{-1}$ indicate a phase transition. The linear extrapolants converge rapidly, suggesting $x_c=1.6\pm 0.3$. The convergence of the slopes is slow, suggesting that a power-behaved gap is not the best *Ansatz*. For the *O(3) model* the ratios of $[F(x)]^{-1}$ show no evidence of a phase transition. The linear extrapolants decrease with increasing l, suggesting a singularity at $x\to\infty$. The O(4) model behaves similarly to the O(3) model, but rules even more decisively against a phase transition at finite x.

It is interesting to pursue the observation that the O(2) model series for F does not favor a power-law singularity. Kosterlitz[8] has predicted that the correlation length ξ should have an essential singularity,

$$\xi \underset{T\to T_c}{\simeq} \exp[b/(T-T_c)^{1/2}], \tag{14}$$

for the theory formulated on a square lattice. So, consider the possibility that the mass gap vanishes as

$$F \underset{x\to x_c}{\simeq} \exp[b'(x_c-x)^{-\sigma}], \tag{15}$$

which would imply

$$\beta(g)/g \underset{x\to x_c}{\simeq} (2b'\sigma x_c)^{-1}(x_c-x)^{1+\sigma}. \tag{16}$$

To test Eq. (1) we extrapolate our series for β to the critical point using a [4, 4] Padé approximant,

$$\beta(g)=\left(\frac{1-0.1413x-0.2589x^2-0.1662x^3+0.9704x^4}{1+1.8587x+0.9584x^2+0.1664x^3-0.07654x^4}\right)g, \tag{17}$$

which is plotted in Fig. 2. It has a zero at $x_c=1.7$ in agreement with the ratio tests and a good power-law fit to the curve is obtained with

$$\beta(g)/g \simeq 0.232(x_c-x)^{1.5}, \tag{18}$$

which gives $\sigma=0.5$ as suggested by Ref. 8. This detailed numerical agreement is probably fortuitous (it depends on the precise procedure used here), but both Fig. 2 and the ratio test suggest that the β function has curvature in the critical region. Higher-order calculations may be worth pursuing.

Finally, consider the O(3) model. We want an

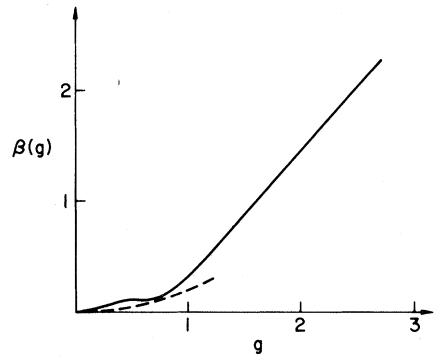

FIG. 3. β function of the O(3) model. The solid curve is obtained from a Padé extrapolation from strong coupling and the dashed curve is the weak-coupling result.

interpolation formula for its β function which satisfies the boundary conditions (1) $\beta(g)\sim g$ for large g; (2) $\beta(g)\sim g^2$ for small g. Item (1) follows from Eq. (10) and item (2) follows from ordinary perturbation theory[9] which gives $\beta = (2\pi)^{-1}g^2 + (4\pi^2)^{-1}g^3 + \dots$. We can fit these boundary conditions by forming the [2, 3] Padé approximant to the series for $[\beta(g)/g]^2$. The resulting curve is shown in Fig. 3. It has the following features: (1) The Padé approximant accurately matches onto the weak-coupling curve in the intermediate coupling region. This is a nontrivial result which provides more evidence against a phase transition at nonzero g. (2) The intermediate-coupling region is very narrow. The curve is essentially linear down to $g\simeq\frac{3}{4}$ where the perturbative result should take over.

Similar analyses have been carried out for the O(4) model. The matching between the Padé approximant and weak-coupling perturbation theory is even more precise and the transition from weak to strong coupling occurs closer to the origin ($g\simeq\frac{1}{2}$). Thus, we find no striking difference between the two non-Abelian models.

A more detailed study of these models will appear elsewhere.

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