Renormalization-Group Approach to the Percolation Properties of the Triangular Ising Model

W. Klein, H. Eugene Stanley, Peter J. Reynolds, and A. Coniglio[(a)]
Physics Department, Boston University, Boston, Massachusetts 02215
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We present a renormalization-group approach for treating the percolation properties of the nearest-neighbor triangular Ising model. We obtain exponents for the line of percolation transitions $T_c \leq T \leq \infty$. In particular, we find a possibly exact result for the connectedness-length exponent $\nu_p = \ln\sqrt{3}/\ln(\frac{3}{2})$ in the "pure" percolation ($T=\infty$) limit, which holds for $T_c < T \leq \infty$. At $T=T_c$ we find the connectedness-length exponents of the percolation problem to be identical to the correlation-length exponents for the thermal problem.

The percolation problem has been an active area of research for many years. The properties of clusters formed by sites occupied at random has been studied extensively. In contrast, there has been relatively little effort made to understand the more difficult, but physically more interesting, properties of a model where the site occupation is correlated. Such models are relevant, e.g., in studying temperature effects in polymer gelation, as well as the effects of quenched impurities in random magnets in which the quenching was done at a finite temperature. We shall also see that the connectivity properties of such models may be relevant to the understanding of *thermal* phase transitions.

A correlated percolation problem that has received some attention is the study of the connectivity properties of the lattice gas or Ising model[1-3]. In this model, spins couple via a nearest-neighbor Ising interaction and with an external magnetic field. The spin-up state is associated with an empty site, and the spin-down state with an occupied site.[1-3] Little information exists about the critical exponents[2] associated with the percolation transitions of the lattice gas except at $T=\infty$, which corresponds to random, or pure, percolation.

Here we describe a renormalization-group (RG) procedure for treating the correlated percolation problem on the triangular lattice. In order to define our RG, we construct an operator $\Theta(s)$, a function of the spin variables $\{s_i\}$, where $\Theta(s)$ gives the number of clusters for a given configuration of spins. For example, in one dimension,[4] $\Theta(s) = \frac{1}{2}\sum_i (1+S_i)$.

The average number of clusters, which in percolation plays the role of the free energy,[5] is

$$G_P(h,K) = \langle\Theta(s)\rangle. \tag{1a}$$

We write $\langle\Theta(s)\rangle$ as

$$\langle\Theta(s)\rangle = \frac{\partial F(h,K,\omega)}{\partial\omega}\bigg|_{\omega=0}, \tag{1b}$$

where $F(h,K,\omega) = \ln Z(h,K,\omega)$, and

$$Z(h,K,\omega) = \sum_{\text{config}} \exp[\mathcal{H}(s) + \omega\Theta(s)]. \tag{1c}$$

Here $\mathcal{H}(s)$ is the usual nearest-neighbor Ising Hamiltonian, $\mathcal{H}(s) = h\sum_i s_i + K\sum_{\text{nn}} s_i s_j$, where h and K are the dimensionless magnetic field and coupling constant.

Equation (1c) has the form of a thermal partition function, since $\Theta(s)$ can be writtein as the sum of products of spin variables. The singularities of the function $G_P(h,K)$ must be contained in $Z(h,K,\omega)$. Since the singularities of $Z(h,K,\omega)$ are associated with an infinite length, we can apply the RG techniques of thermal critical phenomena. In particular, we employ a Kadanoff block-spin transformation.[6] Note that $\Theta(s)$ has the form

$$\Theta(s) = \sum_a K_a s_a, \tag{2}$$

where, in the notation of Ref. 7, K_a represents all possible coupling constants (two-spin, three-spin,...; nearest-neighbor, next-nearest neighbor,...), and s_a represents the corresponding spin products. We perform the partial trace

$$\exp[\mathcal{H}_{\text{eff}}'(s')] = \sum_s P(s,s')\exp[\mathcal{H}_{\text{eff}}(s)], \tag{3}$$

where $P(s,s')$ is the RG weight function[7] and $\mathcal{H}_{\text{eff}} = \mathcal{H}(s) + \omega\Theta(s)$.

An important simplification for the triangular lattice is that the singular part of $G(h,K)$ is obtained only from $\Theta_e(s)$, the even part of $\Theta(s)$, where we write $\Theta(s) = \Theta_e(s) + \Theta_o(s)$ with $\Theta_e(-s) = \Theta_e(s)$ and $\Theta_o(-s) = -\Theta_o(s)$. Specifically, it is straightforward to show, following the argument of Sykes and Essam,[8] that $\langle\Theta_o(s)\rangle \equiv G_o(h,K)$ has a contribution only from clusters of size smaller than or equal to 3, and therefore can play no role in the percolation singularity. Hence, for the purpose of studying the *percolation transitions*, we need only consider

$$G_e(h,K) = \langle\Theta_e(s)\rangle = \left[\frac{\partial}{\partial\omega}\ln\Big\{\sum_{\rm config}\exp[\mathcal{H}(s)+\omega\Theta_e(s)]\Big\}\right]_{\omega=0}. \qquad (4)$$

We now have a "Hamiltonian" that consists of a nearest-neighbor interaction K [from $\mathcal{H}(s)$] and interactions [from $\omega\Theta_e(s)$] which are even under spin inversion, and have arbitrarily small unrenormalized coupling constants.

In addition to the usual restrictions[7] placed on the weight function $P(s,s')$, it must also satisfy two percolation criteria. First, it must reflect the connectivity aspects of the problem. We choose $P(s,s')$ to be the "connectivity rule"[9]: A block spin is considered to be "down" (i.e., cell occupied) when there exists a connected path of "down" spins spanning the cell.[10] The second criterion that $P(s,s')$ must satisfy is that it conserve the symmetry of $\mathcal{H}(s)$ and $\Theta_e(s)$. We adopt this rule because the symmetry of both $\mathcal{H}(s)$ and $\Theta_e(s)$ could play an important role in the form of the percolation singularity.

It is not at all obvious that a weight function can be chosen that satisfies both criteria. Thus far we have found one, and only one, such weight function. It is for the triangular lattice with a three-spin cell [Fig. 1(a)], and is given by

$$P(s,s') = \tfrac{1}{2}[1+\tfrac{1}{2}s'(s_1+s_2+s_3-s_1s_2s_3)], \qquad (5)$$

which is identical to the majority rule of Niemeijer and van Leeuwen.[11] This $P(s,s')$ is the connectivity rule and clearly conserves the spin inversion symmetry of $\Theta_e(s)$ and $\mathcal{H}(s)$. Therefore the $P(s,s')$ chosen above satisfies both criteria.

We note that with this Hamiltonian and weight function, our RG is identical to a $d=2$ Ising *thermal* problem. Hence we can use all of the results that have been obtained for such models.[12]

We consider three cases: $K=0$, $0<K<K_c$, and $K=K_c$. We begin with $K=0$, which describes pure percolation. For this case, our "Hamiltonian" reduces to $h\sum_i s_i + \omega\Theta_e(s)$. Since ω is arbitrarily small, we know from extensive work of others[7] that the fixed point that controls the physics of this "Hamiltonian" is ($\omega=0$, $h=0$). Moreover, we know from the same work that all eigenvalues in the even direction [i.e., thos that are associated with $\Theta_e(s)$] are irrelevant. At this fixed point, there is one relevant eigenvalue associated with the field[7] h,

$$\lambda_h \equiv b^{\tilde{y}_h} = \tfrac{3}{2}. \qquad (6)$$

We expect that the free energy $F(h,K=0,\omega)$ has the form

$$F(h,K=0,\omega) = A(h,\omega)F_s(h,K=0,\omega) + B(h,\omega), \qquad (7)$$

where A and B are analytic functions of h and ω, and F_s is the singular part of F. The function F_s has the usual scaling form in the neighborhood of $h=\omega=0$,

$$F_s(h,K=0,\omega) = h^{d/\tilde{y}_h} f(\omega h^{\tilde{x}}). \qquad (8)$$

Since the eigenvalues associated with $\Theta_e(s)$ are irrelevant, it follows that $\tilde{x}>0$. From Eqs. (4), (7), and (8), we have

$$G_s(h,K=0) \sim A' h^{d/\tilde{y}_h}, \qquad (9)$$

where G_s is the singular part of G_e, and $A' \equiv [\partial A(h,\omega)/\partial\omega]_{h=\omega=0}$. With the assumption that $A'\neq 0$, together with the relation $p-\frac{1}{2}=\frac{1}{2}\tanh h$

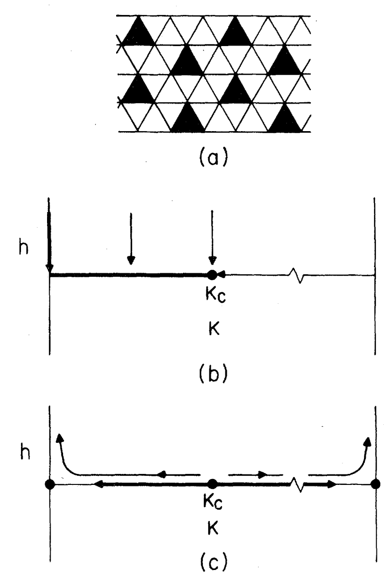

FIG. 1. (a) Triangular lattice with three-spin cells. (b) Phase diagram indicating the percolation transitions on a two-dimensional triangular lattice. The line $0\le K\le K_c$ is a line of percolation points. The arrows indicate the paths along which the percolation points are approached. (c) Schematic diagram illustrating RG flow lines in the h-K plane. At $h=0$, for $K<K_c$, the flow is to the $K=0$ fixed point, while for $K>K_c$, the flow is to the $K=\infty$ fixed point, and at $K=K_c$ the flow, in the higher-dimensional coupling-constant space (not shown), is to the Ising fixed point.

$\simeq \frac{1}{2}h$ for small h, we can calculate $2-\alpha_p$ and, with hyperscaling, the connectedness-length exponent ν_p, defined by $\xi_p \sim (p-p_c)^{-\nu_p}$. Here ξ_p is, for percolation, the quantity analogous to the correlation length ξ_T in thermal phenomena. We find $p_c = \frac{1}{2}$, which is known to be exact.[8] Also, we find

$$2-\alpha_p = d\tilde{y}_h^{-1} = d\ln b/\ln\lambda_h$$
$$= \ln 3/\ln(\tfrac{3}{2}) = 2.7095, \ldots \qquad (10a)$$

and

$$\nu_p = \tilde{y}_h^{-1}\ln\sqrt{3}/\ln(\tfrac{3}{2}) = 1.3547, \ldots, \qquad (10b)$$

where $b=\sqrt{3}$ is the rescaling length. This value of ν_p is in excellent agreement with the value[13] $\nu_p = 1.356 \pm 0.015$ (site) and with the preliminary value[14] 1.365 ± 0.015 (bond) obtained from RG calculations using very large cells. One should also note the series values[15] $\nu_p = 1.32^{+0.02}_{-0.07}$ (site), $\nu_p = 1.34 \pm 0.02$ (bond), and $2-\alpha_p = 2.668 \pm 0.004$ (bond).

There are two aspects of this result that merit emphasis. One is that the calculation of λ_h requires *no approximations*. Therefore, if the arguments and assumptions are correct, Eq. (10) is *exact*. In this context, we note that the result quoted in Eq. (10) has appeared in the literature,[9] with, however, no claim to being exact. In Ref. 9, there was no criterion for distinguishing this result from different results obtained on other lattices. There is also no way for such approximate RG's to distinguish between different results obtained with different cells on the same lattice. In this work, we have presented two criteria which, to the best of our knowledge, only the three-spin cell on the triangular lattice satisfies. Therefore we believe Eq. (10) to be an exact result, although we have no rigorous proof.

The second point we wish to emphasize is that the eigenvalue that determines the exponent ν_p corresponds to a scaling field that is redundant[16] in the Ising thermal problem. This redundancy is associated with the fact that different elements of the class of weight functions that one can use for the Ising thermal problem give different values of λ_h at the $T=\infty$ fixed point. We have, by adopting the connectivity rule, limited our choice of weight functions to only one of this class, and thereby removed this degeneracy for the *percolation* problem.

Thus far, we have treated pure percolation. The second case we will consider is correlated percolation, with $0<K<K_c$. Since ω is arbitrarily small, we expect from the RG analysis of the Ising model[7] that in this range of K, the $K=0$ fixed point again determines the physics. The trajectories for $h=0$ [Fig. 1(c)] flow into the noninteracting fixed point. This implies that the line $(h=0,\ K<K_c)$ is a line of "connectivity critical points" with the same exponents as for pure percolation.

Finally, we consider the point $(h=0,\ K=K_c)$. Since ω is arbitrarily small, we expect that the Ising critical fixed point determines the physics for *percolation* at $K=K_c$. As for the $K=0$ case, we expect $F_s(h, K=K_c, \omega)$ to have a scaling form

$$F_s(h, K=K_c, \omega) \sim h^{d/\hat{y}_h}\hat{f}(\omega h^{\hat{x}}), \qquad (11)$$

where $\hat{x}>0$ if the operator $\Theta_e(s)$ has no component in the relevant even direction, and $\hat{x}<0$ if it does. The $\Theta_e(s)$ operator is, of course, extremely complicated, and a calculation of the eigenvalues conjugate to the scaling fields associated with $\Theta_e(s)$ has not been done. However, we can argue that if $\hat{x}<0$, the exponent for ξ_p would violate a rigorous inequality.[3] For $\hat{x}<0$, the dominant singularity of $G_p(h, K=K_c)$ becomes

$$G_s(h, K=K_c) \sim h^{(d/\hat{y}_h+\hat{x})}. \qquad (12)$$

Assuming hyperscaling, we have

$$\xi_p \sim h^{-(1/\hat{y}_h+\hat{x}/d)} \equiv h^{-\hat{\nu}}. \qquad (13)$$

Since $\hat{x}<0$, $1/\hat{y}_h+\hat{x}/d < 1/\hat{y}_h$, violating the relation tion[3] $\hat{\nu} \geq 1/\hat{y}_h$. Therefore, to satisfy this rigorous inequality, we must have $\hat{x}>0$. With this result, Eq. (11) leads to [cf. Eq. (7) and following]

$$G_s(h, K=K_c) \sim h^{d/\hat{y}_h}. \qquad (14)$$

We could also write a scaling form for $F_s(h=0, K=K_c, \omega)$ which leads, by arguments similar to those used above, to the result

$$G_s(h=0, K-K_c) \sim (K-K_c)^{d/\hat{y}_T}. \qquad (15)$$

We stress that $\hat{y}_h$ and $\hat{y}_T$ are the scaling powers associated with the *ferromagnetic Ising thermal* fixed point. This result, together with hyperscaling, implies that asympotocially the connectedness length is proportional to the correlation length for the thermal problem:

$$\xi_p/\xi_T \sim \text{const.} \qquad (16)$$

Thus, for the triangular lattice, percolation clusters and Ising droplets can be described by characteristic lengths having identical critical exponents.

In summary, the RG approach presented above has led to three new results: (i) For pure percolation, $2-\alpha_p = 2\nu_p = \ln 3/\ln\frac{3}{2}$ [Eq. (10)] is probably an exact result, and is associated with an eignevalue which is redundant in the Ising *thermal* problem; (ii) the line ($h=0$, $T>T_c$) of Fig. 1(b) is a line of percolation points with the same exponents as pure percolation; (iii) at the point ($h=0$, $T=T_c$), two of the three scaling powers ($\hat{y}_h$, $\hat{y}_T$) that determine the percolation exponents are identical to the two scaling powers (y_h, y_T) associated with the nearest-neighbor Ising thermal critical point. The implication of this result is that in the triangular lattice, critical droplets are described by the same characteristic length as percolation clusters.

Although the above arguments were perforce limited to the triangular lattice, one expects from universality considerations that the exponents determined above are the same for any two-dimensional lattice.

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[(a)]Permanent address: Gruppo Nazionale di Struttura della Materia, Istituto di Fisica, Università di Napoli, Napoli, Italy.

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