New Theoretical Prediction of the Ground-State Hyperfine Splitting in Muonium

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We have computed a new contribution to the ground-state hyperfine splitting in muonium: $-2(\alpha/\pi)^2(m/M)[\ln(M/m)]^2 E_F = -6.6$ kHz. Uncalculated theoretical contributions are expected to be no larger than a few kHz. The new theoretical estimate, $\nu_{\rm theor} = 4\,463\,297.9(7.0)$ kHz is in good agreement with experiment: $\nu_{\rm expt} = 4\,463\,302.35(0.52)$ kHz. The bulk of the uncertainty in the theory is due to uncertainty in the measured value of μ_μ/μ_p.

High-precision measurements of the ground-state hyperfine splitting (hfs) in muonium (μ^+e^-) allow a detailed test of our understanding of two-body bound states in quantum field theory and particularly in quantum electrodynamics. The current experimental value is[1]

$$\nu_{\rm expt} = 4\,463\,302.35(0.52)\ \text{kHz}.$$

Until recently, theoretical predictions were known only to within 10–15 kHz.[2] The bulk of this uncertainty comes from three sources: (1) Uncalculated terms of $O((\alpha^2 m/M)[\ln(M/m)]E_F \sim 6\ \text{kHz})$ coming from two-loop ladder and cross-ladder diagrams ($E_F \sim \frac{8}{3}\alpha^4 m^2/M$ is the lowest-order hfs); (2) uncalculated terms of $O((\alpha/\pi)^2(m/M)[\ln(M/m)]^2 E_F \sim 3\ \text{kHz})$; (3) uncertainty in the measured value of μ_μ/μ_p leading to possible errors of ± 5 kHz.[1]

In a new paper, Bodwin, Yennie, and Gregorio have demonstrated that terms of the first sort cancel completely.[3] In this Letter, we describe a calculation of all terms of the second type—the largest remaining contributions to the muonium hfs. These arise from radiative corrections to the electron and photon lines in the one-loop ladder diagrams. We find a total contribution of

$$\Delta E = -2\left(\frac{\alpha}{\pi}\right)^2 \frac{m}{M}\left(\ln\frac{M}{m}\right)^2 E_F = -6.6\ \text{kHz}. \tag{1}$$

Consequently the current theoretical prediction is

$$\nu_{\rm theor} = 4\,463\,297.9(7.0)\ \text{kHz}.$$

The agreement with experiment is excellent. The major source of error is now in μ_μ/μ_p, and improved values for this constant will be available in the near future. There remains a theoretical uncertainty of a few kHz due to uncalculated terms of the form[4]

$$(\alpha/\pi)^2(m/M)[\ln(M/m)]E_F \sim 0.6\ \text{kHz},$$

$$(\alpha^2 m/M)E_F \sim 1.1\ \text{kHz}, \quad (\alpha^3/\pi)E_F \sim 0.6\ \text{kHz}.$$

In what follows, we first review the calculation of the $O((\alpha m/M)[\ln(M/m)]E_F)$ terms in the hfs.[5] Building upon this analysis, we than compute all $O((\alpha^2 m/M)[\ln(M/m)]^2 E_F)$ contributions. Finally we present a very simple argument supporting the conclusions of Bodwin, Yennie, and Gregorio.[3]

The terms of $O((\alpha m/M)[\ln(M/m)]E_F)$ come from one-loop ladder graphs (Fig. 1). Factors of $\ln(M/m)$ can only arise from the integration region $m \ll |k| \ll M$, because only in this region are the integrals sensitive to both mass scales m and M. Thus we can restrict the integration so that $|k| \geq m$. This is useful for three reasons: (1) It prevents double counting of lower-order (in α) contributions coming from the nonrelativistic region. (2) The relative momenta in the wave functions (i.e., p, q), being nonrelativistic ($\sim \alpha m$), can be neglected in the kernel, and the integration over wave functions may be trivally performed:

$$\left|\int d^3p\, \varphi(\vec{p})\right|^2 = |\varphi(\vec{x}=0)|^2 \simeq \alpha^2 m^3/\pi. \tag{2}$$

(3) The effects due to binding are negligible ($O(\alpha^2)$ corrections) and the external legs can be put on mass shell. Consequently the contribution from this region of momentum space in the kernels of Fig. 1 is gauge invariant.[6] Thus the $O((\alpha m/M)\ln(M/m)E_F)$ hfs due to the uncrossed ladder [Fig. 1(a)] is (in Feynman gauge)

$$\frac{|\varphi(0)|^2}{3}\frac{ie^4}{(2\pi)^4}\int\frac{d^4k}{k^6}\,\frac{\mathrm{Tr}^{(e)}[\gamma^i\gamma_5\gamma^\mu \not{k}\gamma^\nu \frac{1}{2}(1+\gamma_0)]\,\mathrm{Tr}^{(\mu)}[\gamma^i\gamma_5\gamma_\mu[-\not{k}+M(\gamma^0+1)]\gamma_\nu\frac{1}{2}(1+\gamma_0)]}{k^2-2Mk^0},$$

where we have projected out the hyperfine interaction (spin-spin) using

$$\tfrac{1}{3}\,\mathrm{Tr}[\gamma^i\gamma_5\mathfrak{M}^{(e)}\tfrac{1}{2}(1+\gamma_0)]\,\mathrm{Tr}[\gamma^i\gamma_5\mathfrak{M}^{(\mu)}\tfrac{1}{2}(1+\gamma_0)]. \tag{3}$$

It is really demonstrated that the cross-ladder graph [Fig. 1(b)] gives an identical contribution, and therefore the sum of the two is

$$\Delta E_{\mathrm{L+CL}} = \tfrac{16}{3}\,|\varphi(0)|^2 \frac{ie^4}{(2\pi)^4}\int_m \frac{d^4k}{k^6}\,\frac{\vec{k}^2+3k^2}{k^2-2Mk^0}.$$

Performing a Wick rotation ($k^0 = ik\cos\varphi$, $|\vec{k}| = k\sin\varphi$) and symmetrizing in k^0, we finally obtain

$$\Delta E_{\mathrm{L+CL}} = \frac{\alpha}{\pi}\frac{m}{M}E_F\frac{2}{\pi}\int_m^{\sim M}\frac{dk}{k}\int_0^{\pi}d\varphi\sin^2\varphi[3+2\tan^2\varphi]\,\theta(|k\cos\varphi|-m)$$

$$\simeq -3\frac{\alpha}{\pi}\frac{m}{M}E_F\int_m^{\sim M}\frac{dk}{k} = -3\frac{\alpha}{\pi}\frac{m}{M}\ln\left(\frac{M}{m}\right)E_F, \tag{4}$$

where only logarithmic terms have been retained.

Terms of $O((\alpha^2 m/M)[\ln(M/m)]^2E_F)$ are due to first-order radiative corrections on the electron and photon lines of the graphs just discussed (Fig. 2). When $|k| \gg m$ (and Euclidean), these radiative corrections modify the bare graphs only by an overall factor of the form

$$(K\alpha/\pi)\ln(k^2/m^2), \quad k^2 \gg m^2.$$

Introducing such a factor into the integrand of (4) results in a splitting of

$$\Delta E = -3K\left(\frac{\alpha}{\pi}\right)^2\frac{m}{M}\left(\ln\frac{M}{m}\right)^2E_F + O\left(\left(\frac{\alpha}{\pi}\right)^2\frac{m}{M}\ln\left(\frac{M}{m}\right)E_F\right) \tag{5}$$

As is well known, the constants K for the vacuum-polarization, electron propagation and vertex corrections [Figs. 2(a), 2(b), and 2(c)] are $\frac{1}{3}$, $\frac{1}{4}$, and $-\frac{1}{4}$, respectively.[7] Furthermore, it is readily demonstrated that $K = \frac{1}{4}$ for the radiative correction in Fig. 2(d):

$$T_{\gamma e\to\gamma e} \simeq \frac{-ie^2}{(2\pi)^4}\int_m\frac{d^4q}{q^6}\,\bar{u}(0)\gamma_\mu \not{q}\,\not{\epsilon}_2\frac{1}{\not{q}+\not{k}}\not{\epsilon}_1\not{q}\gamma^\mu u(0)$$

$$\simeq \frac{-ie^2}{(2\pi)^4}\int_m^{\sim k}\frac{d^4q}{q^4}\,\bar{u}(0)\not{\epsilon}_2\frac{1}{\not{k}}\not{\epsilon}_1 u(0) \simeq \frac{\alpha}{4\pi}\ln\left(\frac{k^2}{m^2}\right)T_{\mathrm{Born}}, \quad k^2 \gg m^2.$$

Thus the leading contribution from the kernels in Fig. 2 is just that quoted above [Eq. (1)]:

$$\Delta E = -3[2(\tfrac{1}{3}) + \tfrac{1}{4} + 2(-\tfrac{1}{4}) + \tfrac{1}{4}]\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m}{M}\right)\ln\left(\frac{M}{m}\right)^2E_F.$$

These diagrams are the only source of $O(\alpha^2 \times [\ln(M/m)]^2E_F)$ splittings. Momenta in first-order radiative corrections to the muon line are scaled by M and thus cannot contribute $[\ln(M/m)]^2$ terms. Second-order radiative corrections to the one-photon-exchange interaction result in no $[\ln(M/m)]^2$. The only remaining graphs are the two-loop ladder and cross-ladder graphs considered in Ref. 3. Two of these are illustrated in Fig. 3. Separately these graphs contain terms of

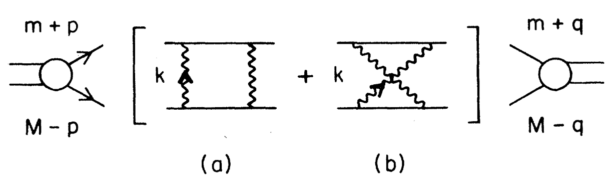

FIG. 1. Kernels contributing to $O(\alpha\,(m/M)\ln(M/m)E_F)$ hfs.

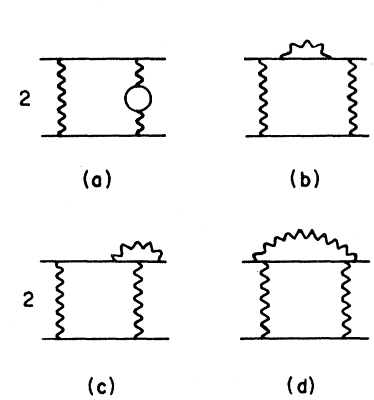

FIG. 2. Kernels contributing to $O(\alpha^2(m/M)[\ln(M/m)]^2 \times E_F)$ hfs.

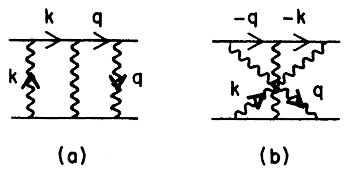

FIG. 3. Two graphs which cancel to $O(\alpha^2(m/M) \times \ln(M/m)E_F)$.

$O((\alpha^2 m/M)[\ln(M/m)]E_F)$. However, one can demonstrate that the graphs cancel in pairs to this order [of course, $O((\alpha^2 m/M)E_F)$ terms may remain] if the following intuitively reasonable assumptions are valid: The external momenta can be set to zero since logarithmic contributions come only from the region of relativistic momenta (as above); and the electron mass can be neglected because logarithmic contributions arise only from the region of momenta $m \ll k, q \ll M$ (in *both* loops).

Bodwin *et al.* have verified these by direct computation. The logarithmic contribution of any diagram is then canceled by that from the diagram obtained by reversing the electron line while leaving all other propagators unchanged (e.g., Fig. 3). This is because the traces [as in Eq. (3)] associated with the electron lines in each diagram are equal but opposite in sign. For example, the electron traces for the diagrams in Figs. 3(a) and 3(b) are, respectively [the γ_0 in $\frac{1}{2}(1+\gamma_0)$ of Eq. (3) obviously does not contribute here],

$$\mathrm{Tr}^{(e)}\left[\gamma^i\gamma_5\gamma_\alpha\frac{1}{\not q}\gamma_\beta\frac{1}{\not k}\gamma_\delta\frac{1}{2}\right],$$

$$\mathrm{Tr}^{(e)}\left[\gamma^i\gamma_5\gamma_\delta\frac{-1}{\not k}\gamma_\beta\frac{-1}{\not q}\gamma_\alpha\frac{1}{2}\right] = -\mathrm{Tr}^{(e)}\left[\gamma^i\gamma_5\gamma_\alpha\frac{1}{\not q}\gamma_\beta\frac{1}{\not k}\gamma_\delta\frac{1}{2}\right].$$

Thus the logarithmic contribution [and indeed all contributions of $O(\alpha^2 E_F)$ from the region $m \ll k, q \ll M$] due to all two-loop ladder graphs cancel completely.

Needless to say, many of the approximations used in this paper are valid only when computing leading logarithmic terms. Nonleading terms must be analyzed within the context of an exact bound-state perturbation theory (see, for example, Refs. 2 and 4). However, it appears likely that the uncertainty due to our ignorance of these terms will for the present be no larger than experimental uncertainties in the relevant physical constants (i.e., μ_μ/μ_p, α).

The immediate stimulus for this work was the analysis by Bodwin, Yennie, and Gregorio described in Ref. 3. We are indebted to them for several conversations. We also thank V. Hughes and R. Horgan for fruitful discussions. This work was supported by the U. S. Department of Energy.

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[5]W. A. Newcomb and E. E. Salpeter, Phys. Rev. 97, 1146 (1955); H. Grotch and D. R. Yennie, Rev. Mod. Phys. 41, 350 (1969).

[6]This is true even though the momentum k flowing in the photon lines is cut off at m since the cutoff is symmetric when $k \to -k$.

[7]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964). These constants are easily computed in much the same manner as the result for Fig. 2(b) is determined in this Letter.