astrophysical collapse. However, it is reasonable to assume that the qualitative picture will be the same. It is unlikely that any significant amount of gravitational wave energy will be emitted during the infall phase of any collapse. One should expect generation of gravitational waves during a bounce, e.g., in the formation of a neutron star.

From this point of view one may expect more gravitational radiation during neutron-star formation than during black-hole formation. Of course, we cannot rule out generation of gravitational waves during black-hole formation via mechanisms which have no analog in the cylindrical case. It is possible that a deformed horizon arises in a highly asymmetric collapse and emits gravitational waves resembling the perturbation modes of a Kerr black hole.[17] Smarr and Eppley[18] studied the coalescense of two black holes and found that the deformed horizon which is formed in that case is not an efficient source of gravitational radiation. Another possible mechanism for enhanced radiation is growth of unstable nonaxisymmetric perturbations leading to breakdown of the collapsing object to two orbiting parts or to a rotating barlike configuration.[19] The results of this study fully support the picture[11] that these modes are the most promising, and probably the only possible, source for generating large amounts of gravitational radiation during collapse to a black hole.

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Corrections to the Muonium Hyperfine Splitting of Order $\alpha^6 (m_e{}^3/m_\mu{}^2) \ln(m_\mu/m_e)$

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We calculate contributions of order $\alpha^6(m_e{}^3/m_\mu{}^2)\ln(m_\mu/m_e)$ to the muonium hyperfine splitting and find that they cancel for each class of kernels with a given number of exchanged transverse photons. A simple explanation for this result is offered.

Measurement of the hyperfine splitting (hfs) in the ground state of muonium currently provides the most stringent test of relativistic two-body bound-state theory. The present experimental value of the triplet-singlet energy difference is $4\,463\,302.35 \pm 0.52$ kHz,[1] and further improvement in this measurement is expected in the near future. Unfortunately, the uncertainty in the theoretical prediction[2] (4 463 304 ± 6 ± 10 kHz) is about an order of magnitude larger. The ± 6-kHz component is really an experimental uncertainty; it reflects our lack of precise knowledge of the muon mass (i.e., magnetic moment). The ±10-kHz uncertainty allows for contributions that are not yet incorporated into the theoretical prediction. It comes from the expectation that there is a term of order $\alpha^2(m_e/m_\mu)\ln(m_\mu/m_e)$ relative to the leading hfs that is comparable in magnitude to the term of relative order $\alpha^2(m_e/m_\mu)\ln\alpha^{-1}$. Here we report that this expected term actually vanishes.

A variety of approaches based on the Bethe-Salpeter equation have been applied to the calculation of higher-order corrections to the hfs.[2,3] The details of these approaches differ greatly, and it is important to find a particularly economical procedure in order to calculate the more complicated correction under consideration here. In the present work, we have developed approaches which greatly simplify the computation of known results and make "almost trivial" the treatment of the terms of relative order $\alpha^2(m_e/m_\mu)\ln(m_\mu/m_e)$. Even these simplified procedures are a bit too lengthy to present in a Letter. Here we shall outline an approach involving the Gross equation[4] and give the results of a detailed calculation based on that approach. Then we offer intuitive justification for a procedure that yields these results much more simply. It should be emphasized that the Gross equation is used here for definiteness. The same arguments can be made for any basic equation and with any gauge for the photons.

Our analysis is based on the Gross equation. This equation was used by Lepage[2] in his treatment of the hfs, but we organize the calculation in a somewhat different manner. We begin by writing the noncovariant muon (particle 2) propagator ($S_F\beta$) as follows:

$$\frac{1}{p_0-\vec{\alpha}_2\cdot\vec{p}-\beta_2(m_2-i\epsilon)}=-2\pi i\delta[p_0-E_2(p)]\Lambda_+^{(2)}(\vec{p})+\frac{1}{p_0-H_2(\vec{p})-i\epsilon}, \qquad (1)$$

where $E_i(p)=(\vec{p}^2+m_i^2)^{1/2}$, $\Lambda_+^{(i)}\equiv[E_i+H_i(\vec{p})]/2E_i$, and $H_i=\vec{\alpha}_i\cdot\vec{p}+\beta_i m_i$. The left-hand side of Eq. (1) is represented graphically by a solid line, while the terms on the right-hand side are represented by a double line and a line with an ×, respectively.

Keeping the first term on the right-hand side of Eq. (1), one finds that the Salpeter equation[5] in the Coulomb ladder approximation reduces to

$$\chi_G=\frac{1}{E-E_2(\vec{p})-H_1(\vec{p})}\Lambda_+^{(2)}(-\vec{p})V_c\chi_G. \qquad (2)$$

Since Eq. (2) is intractable, we replace it with an approximate equation that has no hfs. The solutions are expressible in terms of Dirac-Coulomb wave functions plus $1/m_2$ corrections. They are represented graphically by external lines with the muon line doubled. The "pure Coulomb" contribution to the hfs contained in Eq. (2) is easily recovered by the methods of Ref. 2. It is represented by graph C in Fig. 1. Additional contributions to the hfs arise from (i) the second term of Eq. (1) between Coulomb rungs; (ii) crossed Coulomb lines in place of (i); (iii) exchanged transverse photons; and (iv) radiative-correction–type contributions—e.g., electron or muon self-energies or vertices. Only the first three types are discussed here.

It is found that individual graphs can give spurious *lower*-order contributions in m_2^{-1}; therefore, it is important to keep together graphs related by permutations of connections to the muon leg, e.g., the pair CC in Fig. 1. By adding related contributions *before* integration, we find that important cancellations occur which result in simpler integrals.

The two types of logarithms [$\ln\alpha^{-1}$ and $\ln(m_2/m_1)$] are associated with two different momentum ranges. (We assume that $m_2\gg m_1$ and do *not* attempt to work accurately enough so that we may later let $m_2\to m_1$.) The $\ln\alpha^{-1}$ comes from the low-momentum range $\alpha m_1\lesssim p\lesssim m_1$, while the $\ln(m_2/m_1)$ comes from the intermediate range $m_1\lesssim p\lesssim m_2$.

The contributions of relative order $\alpha^2(m_1/m_2)[\ln\alpha^{-1}$ or $\ln(m_2/m_1)]$ from each set are shown in Table I. We note the following features of this table: (i) The $\ln\alpha^{-1}$ terms arise entirely from a subset of graphs which do not have external Coulomb corrections (with respect to particle 1) beyond those already accounted for in the wave function. (ii) An enlarged set of graphs which includes a given $\ln\alpha^{-1}$ term plus related external Coulomb corrections [e.g., C + CC + CCC or T + 2 × CT + 2(CC-T) + CTC + 2 × CCT] has a net $\ln(m_2/m_1)$ contribution of zero. [Note: this ap-

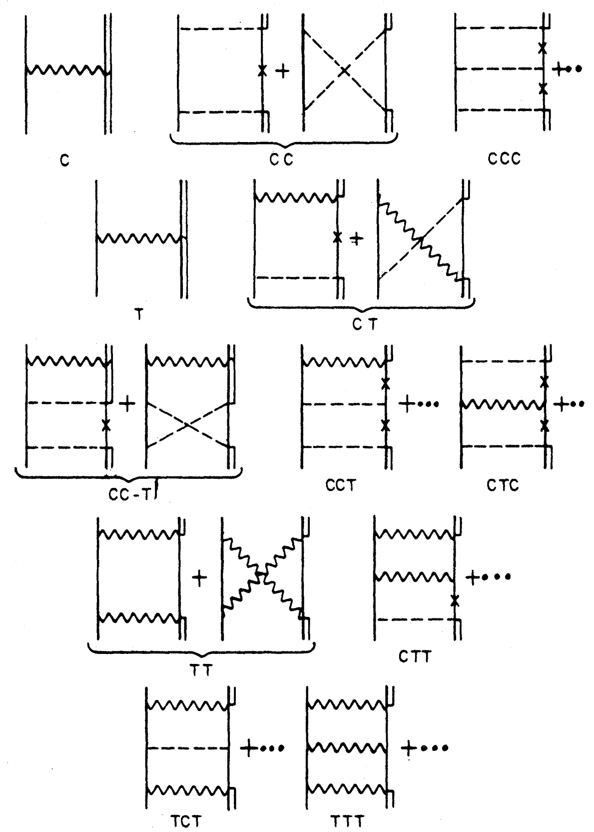

FIG. 1. Graphs contributing to the hfs in relative order $\alpha^2(m_e/m_\mu)$ [$\ln\alpha^{-1}$ or $\ln(m_\mu/m_e)$]. Double lines indicate that the muon is on the positive-energy mass shell, and × lines are the corresponding corrections. "+ ⋯" indicates five additional graphs in which the order of photon emission on the muon leg is permuted. Muon legs internal to a crossed photon structure represent complete muon propagators.

plies only to the given order of α; there *are* terms of relative order $\alpha(m_1/m_2)\ln(m_2/m_1)$.][6] Thus, all contributions of relative order $\alpha^2(m_1/m_2)\ln(m_1/m_2)$ add up to zero.

It is possible to arrive at a simple explanation of this result by grouping related sets of graphs into an enlarged set. One starts by using (2) to add explicit Coulomb factors to some of the original sets so that all graphs in an enlarged set have the same photon connections on the particle-1 leg. These enlarged sets, shown in Fig. 2, now have combinations of three photons (Coulomb or transverse). It turns out that one can obtain the logarithmic content of these graphs by neglecting the small components of the wave function and setting external momenta equal to zero inside the three-photon kernel. At this stage, one can combine graphs so that all doubled lines and × lines join up to give the complete muon propagator.

TABLE I. Contributions of the graphs of Fig. 1 to the piece of the hfs given by $(8/3)\alpha^6(m_e^3/m_\mu^2)[a\ln\alpha^{-1} + b\ln(m_\mu/m_e)]$.

Graphical set	a	b
C	1/4	1/4
CC	0	−1/2
CCC	0	1/4
T	−4	−4
2×CT	0	1
2×CC-T	0	1
2×CCT	0	2
CTC	0	0
TT	9/2	1
2×CTT	0	−1
TCT	5/4	0
TTT	0	0
Sum	+2	0

(The same result can be obtained more directly by using a perturbation approach that has only complete propagators in the kernel.)

Now we consider the integrand of each of these enlarged sets as a function of the energies (k_0, k_0', k_0''; subject to $k_0+k_0'+k_0''=0$) of the exchanged photons. After removing lower-order contributions (in m_2^{-1}), we find that the particle-2 factor has the property of being even (odd) under a simultaneous sign change of all the k_0's for odd (even) number of exchanged transverse photons. In the low-momentum region, the particle-1 integrand has both an even and an odd part, so that the integral is nonvanishing. However, in the intermediate momentum region we may neglect m_1 in the particle-1 propagator, and it is

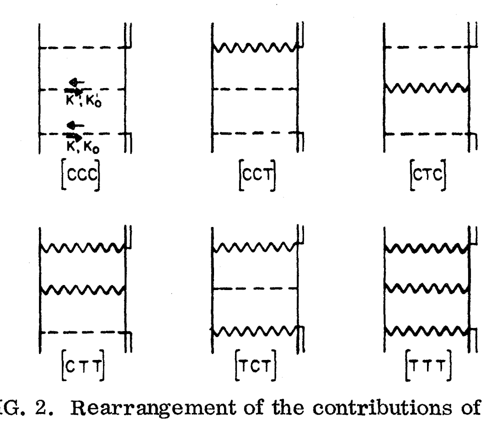

FIG. 2. Rearrangement of the contributions of Fig. 1. Here external momenta are set equal to zero within the kernels. Each set consists of six graphs corresponding to different orders of photon emission.

found that the total integrand is odd. Hence the integration over the intermediate region yields no logarithms.

The vanishing of the contribution from the intermediate region is most easily illustrated in the case of the pure Coulomb correction [CCC]. The six graphs combine to give the muon factor

$$F_\mu = \frac{\vec{\alpha}_2\cdot\vec{k}'\vec{\alpha}_2\cdot\vec{k}}{4m_2^2}\,4\pi i\left\{\delta(k_0)P\left(\frac{1}{k_0'}\right) - \delta(k_0')P\left(\frac{1}{k_0}\right) + \delta(k_0+k_0')P\left(\frac{1}{k_0}\right)\right\}. \tag{3a}$$

This is clearly an odd function of the k_0's. The electron factor common to all graph is

$$F_e = \frac{\vec{\alpha}_1\cdot\vec{k}'\vec{\alpha}_1\cdot\vec{k}}{[(E-m_2+k_0-k_0')^2 - m_1^2 - (\vec{k}+\vec{k}')^2 + i\epsilon][(E-m_2+k_0)^2 - m_1^2 - k^2 + i\epsilon]}, \tag{3}$$

where terms not contributing to the hperfine splitting have been dropped in Eq. (3). In the intermediate range, where $E - m_2 \approx m_1 \to 0$ relative to $\vec{k}$ and $\vec{k}'$, the electron factor becomes an even function of the k_0's, so that the total integrand is odd.[7] Similar results are found for the other graphs of Fig. 2, but the details are more complicated. An even simpler argument leading to this result was mentioned to us by Caswell.[8] He studies the electron factor alone and shows from its symmetry properties that the contributions of different graphs cancel pairwise.

It should be mentioned that the behavior of the contribution of relative order $\alpha(m_1/m_2)\ln(m_2/m_1)$ is different from that just described. This contribution arises from two-photon graphs analogous to the three-photon graphs in Fig. 2. Now, however, the integrand is even in k_0 in the intermediate region yielding a nonvanishing result. In the low-momentum region the integrand does not have the proper structure to develop $\ln\alpha^{-1}$. The terms of relative order $\alpha(m_1/m_2)$ have of course been calculated previously.[6]

The absence of terms of relative order $\alpha^2(m_1/m_2)\ln(m_2/m_1)$ shows that there is no nonanalytic behavior in the mass ratio in relative order α^2 and makes it plausible that these terms can be calculated in a simple way. If this can be done, it would eliminate one of the major obstacles to improving the accuracy of the theoretical prediction. Corrections of higher order in α due to exchanged photons are probably below the present level of experimental accuracy and are certainly less important than uncertainties in corrections due to the insertion of photons into the electron line.[9] Caswell and Lepage[10] have been studying radiative corrections in the intermediate-momentum range and find that they may contribute several kHz. Their paper will contain the simple argument alluded to earlier.

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