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Cylindrical General Relativistic Collapse

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This paper presents results of a numerical study of general relativistic gravitational collapse of infinite cylinders and discusses their implication for possible astrophysical collapse. It is found that almost no gravitational waves are generated during the initial infall period. However, up to 65% of the rest-mass energy is emitted in the form of gravitational waves during the bounce and subsequent oscillations of the cylinder.

Recently there has been a growing interest in measurement of gravitational radiation from astrophysical sources. One of the most promising sources is provided by a star collapsing to a black hole or a neutron star.[1] So far there has been no reliable theoretical calculation of the gravitational radiation emitted by such a collapse and current estimates are based on the quadrupole-moment approximation. This Letter presents results of a numerical study of general relativistic gravitational collapse of infinite rotating cylinders. These are the first fully relativistic calculations of collapse and generation of gravitational waves. Although the infinite cylinders which are discussed here are not realistic astrophysical entities, one can learn a great deal about the basic qualitative features of gravitational wave generation from phenomena which are manifested in their collapse.

I use the Arnowitt-Deser-Misner[2] (ADM) formalism to describe the geometry of space-time. Following Wilson[3] I choose a gauge condition by specifying the shift vector so that the three-metric g_{ij} is diagonal with $g_{rr}=g_{zz}$. (The spatial coordinates are r, z, and φ.) Maximal slicing[4] is imposed as a second gauge condition which determines the lapse function and fixes the trace of the extrinsic curvature $K_j^{\,i}$ to be zero. $g_{\varphi\varphi}$ is determined by the Hamiltonian constraint and $K_r^{\,r}$ and $K_r^{\,\varphi}$ are calculated from the momentum constraints. This method eliminates all hidden degrees of freedom of the Einstein equations. Only two geometric functions ($g_{rr}=g_{zz}$ and $K_\varphi^{\,\varphi}$) are calculated from the standard ADM evolution equations.

Since we are dealing with an idealized configuration it is satisfactory to use perfect fluids with a γ-law equation of state, $p=(\gamma-1)ne$ (n, e, p, and γ are the matter density, internal specific energy, pressure, and adiabatic index, respectively). The hydrodynamic variables are determined from $T^{\mu\nu}{}_{;\nu}=0$. Details of the numerical methods will be discussed elsewhere.[5]

The code was tested for propagation of gravitational Einstein-Rosen[6] wave pulses in vacuum and for implosion and explosion of relativistic cylindrical shock waves. The coupling between the geometric quantities and the matter was checked using the cylindrical energy[7] for static configurations.

Calculations were done for several initial configurations. The adiabatic index γ, mass per unit length, and specific angular momentum were independently varied. The initial configurations were Newtonian (M, the mass per unit length, and $r_{\max}$ satisfied $r_{\max}<e^{4\pi/M}$) and static. The initial density profile consisted of a large core of constant density surrounded by an outer decreas-

ing atmosphere. The pressure was calculated from the hydrostatic equation and the specific entropy was chosen in such a way that the same density profile could by used for different γ.

The collapse was induced by a sudden large reduction of the internal energy. This resulted in a first-order change in the hydrostatic equation, but only a second-order perturbation to the geometric equations. The matter collapsed reaching relativistic velocities. As expected for values of γ greater than 1, the matter bounced and did not collapse to a naked line singularity.[8] After the bounce a shock wave propagated outwards while the massive core oscillated, approaching a steady state. This general hydrodynamic behavior, even for low γ values, resembles the features of spherical collapse with a hard equation of state.[9] The maximum central density, the maximum infall velocity, and the duration of the bounce increased while the bounce radius decreased with a softer equation of state (smaller γ values). Typical variations in the central density ranged between 6×10^2 and 6×10^6. See Table I for details.

No gravitational waves were generated during the initial collapse. However, large pulses of gravitational radiation were generated during the bounce period [see Figs. 1(a)–1(c)]. The total energy of the emitted gravitional waves was highly dependent on the equation of state (see Table I). While less than 1% of the initial rest mass energy was radiated when $\gamma=\frac{7}{6}$, about 65% of the initial rest-mass energy was radiated as gravitational waves when $\gamma=2$. Second and third pulses of gravitational waves carrying substantial amounts of energy were emitted during the oscillation of the core [see Fig. 1(d)].

A decrease in the mass per unit length, which led to a more Newtonian configuration, resulted in a decrease in the efficiency of gravitational wave production. When the mass per unit length was reduced from 0.086 (in units with $G=c=1$) to 0.045 and to 0.01 the efficiency of gravitational wave generation dropped from 35% to 7% and to zero, respectively.

The initial exterior vacuum region had a Levi-Civitâ line element[10]:

$$ds^2=-(r/R)^{4C}dt^2+(r/R)^{-4C(1-2C)}(dr^2+dz^2)+r^2(r/R)^{-4C}d\varphi^2,$$

with $C\sim 2M$ and $R\sim 1$. In the region which I considered, $r<r_{\max}\sim 30$, $(r_{\max}/R)^{-8C^2}\ll 1$, the three-metric was approximately flat. During the infall stage when no gravitational waves were emitted the evolution of $g_{\varphi\varphi}/r^2$ followed that of g_{rr} and the three-metric kept this form. (This depends partially on the gauge choice, $g_{rr}\equiv g_{zz}$.) The C-energy flux followed the inward matter motion. When a gravitational wave package was emitted there was a definite outgoing pulse of C energy (see Fig. 1), accompanied by a sharp deviation

TABLE I. Comparison of a few cases of cylindrical collapse.

Configuration	Initial internal energy multiplied by	γ	Total mass per unit length [a]	Initial central density ρ_i	Maximum central density $\rho_{\max}$	$\rho_{\max}/\rho_i$	Fraction of total energy emitted in first pulse	Fraction of total energy emitted as gravitational waves
Nonrotating	0.1	2	0.086	1.3×10^{-3}	0.73	560	0.35	0.65
Nonrotating	0.1	11/6	0.086	1.9×10^{-3}	1.1	580	0.27	0.43
Nonrotating	0.1	5/3	0.086	1.9×10^{-3}	3	1580	0.22	0.34
Nonrotating	0.1	3/2	0.086	1.9×10^{-3}	13.4	7000	0.17	0.29
Nonrotating	0.1	4/3	0.086	1.9×10^{-3}	102	54000	0.13	0.13
Nonrotating	0.1	7/6	0.086	1.3×10^{-3}	8000	6000000	< 0.01	< 0.01
Nonrotating	0.5	5/3	0.088	1.4×10^{-3}	1.06	760	0.15	0.20
Nonrotating	0.1	5/3	0.046	6.7×10^{-4}	1.3	1940	0.1	0.1
Nonrotating	0.1	5/3	0.01	1.3×10^{-4}	0.3	2300	0.01	0.01
Fast initial rotation	0.1	5/3	0.086	1.3×10^{-3}	0.009	7	0.11	0.17
Slow initial rotation	0.1	5/3	0.046	6.7×10^{-4}	0.32	477	0.18	0.18

[a] Multiply by 4.5×10^{28} for cgs units (g/cm).

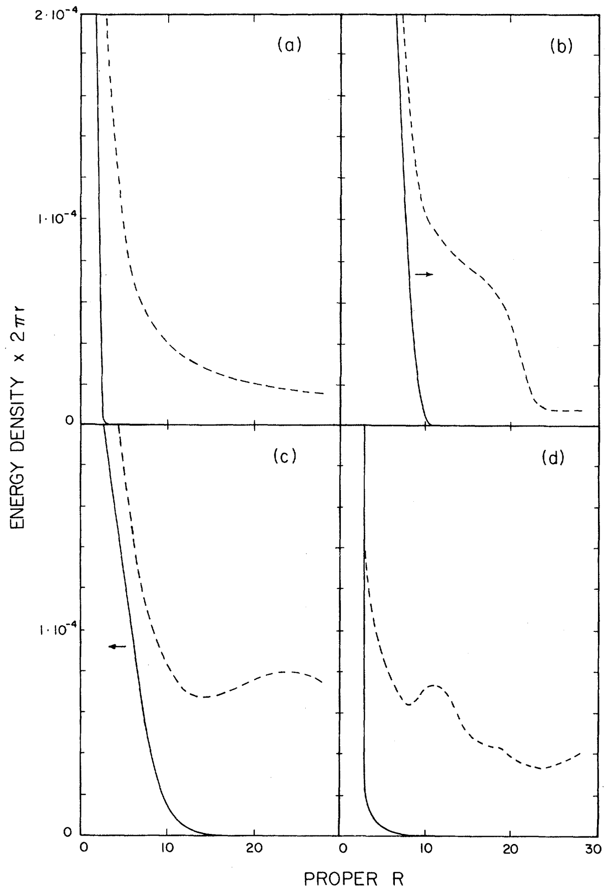

FIG. 1. Matter energy density ρ (solid line) and geometric energy density (dashed line), multiplied by $2\pi r$, as a function of proper distance, for case 3 in Table I. The arrow describes the direction of motion of the matter. (a) First bounce—moment of maximal contraction—no gravitational waves have been generated so far. (b) The first gravitational wave pulse. (c) Propagation of the first graviational wave pulse. (d) Propagation of the second gravitational wave pulse, during oscillations of the core.

from the Levi-Cività form of the metric. As could be expected[7] the final steady-state configuration had a Levi-Cività exterior metric whose R was much smaller than the initial one.

When rotation was added the bounce became shorter and occurred at larger radii. The centrifugal force stiffens the matter and its effect can be compared with an equation of state with $\gamma=2$. (For a homogeneous cylinder $\partial p/\partial r \sim r^{-(2\gamma+1)}$; $F_{grav} \sim M^2 r^{-3}$; $F_{cen} \sim L^2 M^{-1} r^{-5}$, where M and L are the mass and angular momentum per unit length, r is the radius, and F_{grav} and F_{cen} are the Newtonian gravitional and centrifugal forces.) The amount of gravitational waves emitted was larger than in the corresponding nonrotating configuration. A modest amount of angular momentum was most efficient in increasing the production of gravitational waves. High amounts of angular momentum halted the collapse before the matter gathered high enough infall velocity while very small amounts of angular momentum did not have significant effects. The same behavior was observed by Shapiro[11] in Newtonian calculations of spheroidal collapse.

The basic qualitative features of generation of gravitational waves presented here are in correspondence with the quadrupole-moment approximation.[12] In this approximation the gravitational wave luminosity is proportional to $1/\Delta T^6$. Most of the energy is emitted, therefore, during the bounce when ΔT, the time scale for changes in the quadrupole moment, is shortest. Similar behavior was found by Novikov,[13] who considered semiquantitatively Newtonian collapse of spheroids, and by Shapiro[11] and by Saenz and Shapiro,[14] who studied this problem numerically. When comparing their results to those of Thuam and Ostriker[15] for spheroidal dust collapse they found that much more radiation was emitted during the bounce than during the free fall.

The strong dependence of the amount of energy emitted on the adiabatic index might be explained by the shorter bounce period when the equation of state is harder. However, the energy calculated using the quadrupole-moment approximation agreed with the fully relativistic results for the high γ values while it diverged for small γ values. In the latter cases the fully relativistic calculation did not give any significant amount of radiation. Although I could not rule out the possibility that the divergence of the quadrupole-moment approximation was a numerical artifact (the calculations might not be accurate enough for estimating $\dddot{r}$), it seems that there is a strong field effect which suppresses the emission of gravitational waves when the configuration is very relativistic. Post-Newtonian calculations for spheroidal collapse do show lower amounts of gravitational radiation when compared to Newtonian calculations.[16]

The fraction of rest-mass energy which is emitted as gravitational wave energy is suprisingly large (up to 65%). This is a very encouraging figure compared with previous estimates of the efficiency of gravitational wave generation. Clearly, this high efficiency is partially due to the extreme asymmetry of the system and cannot be used as a quantitative estimate for a realistic astrophysical collapse. However, it is reasonable to assume that the qualitative picture will be the same. It is unlikely that any significant amount of gravitational wave energy will be emitted during the infall phase of any collapse. One should expect generation of gravitational waves during a bounce, e.g., in the formation of a neutron star.

From this point of view one may expect more gravitational radiation during neutron-star formation than during black-hole formation. Of course, we cannot rule out generation of gravitational waves during black-hole formation via mechanisms which have no analog in the cylindrical case. It is possible that a deformed horizon arises in a highly asymmetric collapse and emits gravitational waves resembling the perturbation modes of a Kerr black hole.[17] Smarr and Eppley[18] studied the coalescense of two black holes and found that the deformed horizon which is formed in that case is not an efficient source of gravitational radiation. Another possible mechanism for enhanced radiation is growth of unstable nonaxisymmetric perturbations leading to breakdown of the collapsing object to two orbiting parts or to a rotating barlike configuration.[19] The results of this study fully support the picture[11] that these modes are the most promising, and probably the only possible, source for generating large amounts of gravitational radiation during collapse to a black hole.

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Corrections to the Muonium Hyperfine Splitting of Order $\alpha^6 (m_e^3/m_\mu^2) \ln(m_\mu/m_e)$

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We calculate contributions of order $\alpha^6(m_e^3/m_\mu^2)\ln(m_\mu/m_e)$ to the muonium hyperfine splitting and find that they cancel for each class of kernels with a given number of exchanged transverse photons. A simple explanation for this result is offered.

Measurement of the hyperfine splitting (hfs) in the ground state of muonium currently provides the most stringent test of relativistic two-body bound-state theory. The present experimental value of the triplet-singlet energy difference is $4\,463\,302.35 \pm 0.52$ kHz,[1] and further improvement

