Gribov Ambiguities in the U Gauge

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For both Abelian and non-Abelian gauge theories, we find gauge transformations which map fields in the U gauge to other fields in the U gauge. These transformations are not contained in the surviving gauge symmetry after spontaneous breaking (defined as the little group of the vacuum expectation value of the Higgs field). They have both discrete and continuous elements.

Gribov[1] has recently shown that, in a non-Abelian gauge theory, there are several gauge-related Yang-Mills potentials which fulfill the Coulomb gauge (or the Lorentz gauge in four-dimensional Euclidean space). Thus the Coulomb gauge does not determine the potentials uniquely in these theories. The implications of these gauge ambiguities have been studied by Gribov[1] and others.[2]

In this Letter, it will be shown that there are similar ambiguities in the U gauge[3] in both Abelian and non-Abelian gauge theories. These ambiguities can be continuous as well as discrete. As in Gribov's problem, they are also not the same for all field configurations. Their existence throws doubt on the usual heuristic arguments based on the U gauge which lead to the particle interpretation of these theories.[4]

We consider first a gauge theory based on the internal-symmetry group $G = \mathrm{SU}(N)$,[5] Yang-Mills potentials $A_\mu{}^\alpha$ ($\alpha = 1, 2, \ldots, N^2-1$), and a real Higgs multiplet $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{N^2-1})$ which transforms under the adjoint representation $\{g\}$ of $\mathrm{SU}(N)$.[6] Later, we comment on the generalization of these considerations to other gauge groups and representations of the multiplets. If $\{T(\alpha)\}$ are the generators of $\mathrm{SU}(N)$ in, say, the adjoint representation, we can form the matrices

$$A_\mu = A_\mu{}^\alpha T(\alpha), \quad \Phi = \varphi_\alpha T(\alpha). \tag{1}$$

They respond to a gauge transformation as follows[7]:

$$A_\mu \to g A_\mu g^{-1} + (i/e) g \partial_\mu g^{-1}, \qquad \Phi \to g \Phi g^{-1}. \tag{2}$$

Let $\lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_{N^2-1}) \neq 0$ be a space-time independent field which minimizes the Higgs potential (the vacuum expectation value of φ) and let

$$\Lambda = \lambda_\alpha T(\alpha). \tag{3}$$

If the generators are chosen to satisfy

$$\mathrm{Tr}\{T(\alpha)T(\beta)\} = d\delta_{\alpha\beta}, \quad d = \mathrm{const} \neq 0, \tag{4}$$

then the U gauge is defined by[3]

$$\mathrm{Tr}\{\Phi[T(\alpha), \Lambda]\} = 0. \tag{5}$$

Since

$$\mathrm{Tr}\{\Phi[T(\alpha), \Lambda]\} = \mathrm{Tr}\{T(\alpha)[\Lambda, \Phi]\} \tag{6}$$

and since $[\Lambda, \Phi]$ is in the Lie algebra L of $\{g\}$ for which $\{T(\alpha)\}$ is a basis, (5) can be written as

$$[\Phi, \Lambda] = 0. \tag{7}$$

Let G_Λ be the little group of Λ:

$$g\Lambda g^{-1} = \Lambda \quad \text{for } g \in G_\Lambda. \tag{8}$$

Then G_Λ is the "surviving local symmetry" in the gauge (7). The remaining gauge freedoms associated with G_Λ are to be eliminated by further gauge conditions in the customary treatment of the U gauge.

We show below that there are gauge transformations $S \equiv \{s\} \not\subset G_\Lambda$ such that if Φ satisfies (7), then so does $s\Phi s^{-1}$. [The set S in general depends on Φ and Λ.] The gauge conditions suitable for G_Λ do not eliminate these gauge freedoms. The existence of S can change the topology of the set of gauge-inequivalent fields in a fundamental way. Thus, in example 1 below, the gauge-inequivalent φ's are restricted to a half-line and resemble the radial coordinate r in mechanics. In the latter, we know as a consequence that r and its conjugate momentum cannot be quantized like Cartesian coordinates and momenta. Hence, the existence of S throws doubts on the usual quantization procedure in the U gauge and on the particle interpretation based on this gauge. We note that usually canonical quantization is applied not to Φ, but to $\Phi' = \Phi - \Lambda$. Since $\Lambda = O(e^{-1})$,[3] the Φ''s which are related by S differ by terms of the order $O(e^{-1})$. So perturbation theory is not likely to be sensitive to the effects due to S.

Below we will consider in turn the discrete and continuous ambiguities. The considerations which follow now are local. That is, Φ refers to the value of the Higgs field at a given space-time point x. The transformations S are then determined only at x. It is usually assumed in gauge theories that all gauge transformations reduce to the identity map at spatial infinity.[8] Towards the end, we show how these considerations can be globally extended so as to fulfill such boundary conditions. It is likely that the results of this Letter are known to some physicists. Goddard, Nuyts, and Olive[9] have studied the gauge ambiguities due to the Weyl group (see below) in their work on non-Abelian magnetic monopoles. Mack[10] has also studied gauge ambiguities in the U gauge, in particular those associated with generic field configurations.

Discrete ambiguities.—We assume without loss of generality that $\Lambda \in C$, where C is a Cartan subalgebra of L.[11] Now (7) implies that $\Phi \in L_\Lambda$, where L_Λ is the Lie algebra of G_Λ. The Cartan subalgebra of L_Λ can be chosen to be C as well.[12] Thus, there exists $g \in G_\Lambda$ such that $g\Phi g^{-1} \in C$.[11] Since we are interested only in gauge ambiguities not contained in G_Λ, we shall thus assume that $\Phi \in C$.

Let $\mathcal{W} = \{W\}$ be the Weyl group.[13] By definition, $WCW^{-1} \equiv \{WcW^{-1}\}_{c\in C} = C$. Thus, $W\Phi W^{-1} \in C$. Elements of $\mathcal{W}$ not in G_Λ are thus in S. They are new gauge ambiguities.

Example 1.—Let $G = \mathrm{SU}(2)$. Then C is spanned by $T(3)$, say. So $\Lambda = \lambda_3 T(3)$, $\Phi = \varphi_3 T(3)$. With $T(\alpha)_{ij} = -i\epsilon_{\alpha ij}$, the Weyl group $\mathcal{W}$ is $\{e, W = \exp[i\pi T(2)]\}$. The element W is not contained in $G_\Lambda = \mathrm{U}(1)$ whose generator is $T(3)$. We have

$$W\Phi W^{-1} = -\Phi. \tag{9}$$

Thus Φ and $-\Phi$ are gauge related. This gauge ambiguity can be removed by requiring $\varphi_3 \geq 0$.[14]

Example 2.—Let $G = \mathrm{SU}(3)$. Then C is spanned by $T(3)$ and the hypercharge Y, say. So $\Lambda = \lambda_3 T(3) + \lambda_Y Y$, $\Phi = \varphi_3 T(3) + \varphi_Y Y$. The Weyl group $\mathcal{W}$ has six elements e, W_{12}, W_{23}, W_{31}, W_{123}, W_{123}^2. (The notation is that of Schechter, Ueda, and Okubo.[13]) If $\varphi_Y = 0$, then the orbit O_Φ of Φ under $\mathcal{W}$ consists of six points. This is the generic situation[15] in that most points of C have a six-point orbit under $\mathcal{W}$. If $\varphi_3 = 0$, then $W_{12}\Phi W_{12}^{-1} = \Phi$, so that O_Φ consists of three points. This is the nongeneric situation.[15]

If Λ is a generic element,[15] say $\Lambda = \lambda_3 T(3)$, then $G_\Lambda = \mathrm{U}(1) \otimes \mathrm{U}(1)$ and $G_\Lambda \cap \mathcal{W} = \{e\}$. Hence in this case, there are either six or three configurations of fields (at each x) which are gauge related under the full group SU(3), but not under G_Λ.

If Λ is a nongeneric element,[15] say $\Lambda = \lambda_Y Y$, then $G_\Lambda = \mathrm{U}(2)$ with generators $T(\alpha)$ ($\alpha = 1, 2, 3$) and Y. Now $G_\Lambda \cap \mathcal{W} = \{e, W_{12}\}$.[5] If Φ has a six-point orbit O_Φ, W_{12} connects pairs of points of O_Φ; so the new gauge ambiguity is threefold. If on the other hand, Φ is $\varphi_Y Y$ and O_Φ has three points, W_{12} leaves Φ invariant and connects the other two points; so the new gauge ambiguity is twofold.

Continuous ambiguities.—Case 1: The field configuration $\Phi = 0$ and all its transforms by the full gauge group G fulfill (7). Also G acts nontrivially on A_μ [cf. Eq. (2)]. Therefore elements of G not in G_Λ in fact represent new gauge degrees of freedom for field configurations with $\Phi = 0$. Case 2: In Case 1, G was the little group of the null Higgs field. More generally, we can consider the little group G_Φ of a Higgs field $\Phi \neq 0$. If Φ is generic, G_Φ is generated by the Cartan subalgebra C and hence $G_\Phi \subseteq G_\Lambda$. However, if Φ is nongeneric, G_Φ need not be contained in G_Λ and can give rise to new gauge ambiguities. For example, for SU(3), if Λ is generic [say, $\Lambda = \lambda_3 T(3)$] and Φ nongeneric (say, $\Phi = \varphi_Y Y$), then $G_\Lambda = \mathrm{U}(1) \otimes \mathrm{U}(1)$ and $G_\Phi = \mathrm{U}(2)$. Thus, there is a two-parameter family of new gauge freedoms.

Other groups and representations.—The discrete ambiguities above were caused by the Weyl group. Since the latter is well defined for any semisimple group G, such ambiguities are expected to be present whenever φ transforms under the adjoint representation of a semisimple group. It is also present in the Abelian Higgs model [cf. Abers and Lee,[3] p. 20] with φ and $-\varphi$ in the U gauge being gauge related. The situation is similar to the SU(2) example above.

The continuous gauge ambiguities associated with the null Higgs field are as a rule always present in the U gauge. Furthermore, the ambiguities due to the little group of a nonzero Higgs field (Case 2 above) are expected to be present in many instances. The set S for such ambiguities may in general have disconnected components. A generic analysis of these ambiguities seems to require detailed group theory.

Global aspects.—The field φ and the set S have until now been described only locally. We now show how they can be defined globally so as to fulfill the boundary condition $S=\{e\}$ at spatial infinity ($|\vec{x}|\to\infty$). (This would then also mean that the nontrivial elements in S are not global transformations.) To be specific, let $G=SU(3)$ and,

$$\Lambda=\lambda_3 T(3),\quad \Phi=\varphi_3 T(3),\quad 0\leq|\vec{x}|\leq r_1, \tag{10}$$

at any given time. (The definition of Φ for all $\vec{x}$ is given below.) Then the gauge transformation s which equals W_{12} say for $|\vec{x}|\leq r_1$ generates a gauge ambiguity. Since SU(3) is connected, s can be extended globally (consistent with continuity requirements in $\vec{x}$) such that $s=e$ for $|\vec{x}|\geq r_2 > r_1$. We have yet to define Φ for all $\vec{x}$. When we do so, we must make sure that s does not map this Φ out of the U gauge in the region $r_1<|\vec{x}|<r_2$. For this, we can consider those $\Phi\in C$ with $\Phi=0$ when $r_1\leq|\vec{x}|\leq r_2$. [Continuity conditions at $|\vec{x}|=r_1$ cause no difficulty since the choice of φ_3 is at our disposal. Boundary conditions on Φ at infinity (such as $\Phi\to\Lambda$ as $|\vec{x}|\to\infty$) can also be satisfied by a suitable choice of Φ.]

Such considerations are readily generalized.

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[1]V. N. Gribov, Lecture at Proceedings of the Twelfth Winter School of the Leningrad Nuclear Research Institute, 1977 (unpublished).

[2]C. M. Bender, T. Eguchi, and H. Pagels, to be published; R. Jackiw, I. Muzinich, and C. Rebbi, to be published, and references therein.

[3]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973); S. Weinberg, Phys. Rev. D 7, 1068 (1973), and references therein.

[4]For an ambiguity-free gauge (which also does not assume any asymptotic condition for the gauge group), see A. P. Balachandran, A. Stern, Per Salomonson, and Bo-Sture Skagerstam, Syracuse University Report No. SU-4211-107, 1977 (to be published).

[5]Since the center Z_N of SU(N) acts trivially on $A_\mu{}^\alpha$ and φ, we are effectively considering SU(N)/Z_N.

[6]The presence of more multiplets does not invalidate the arguments.

[7]By abuse of notation, we do not distinguish the gauge and internal-symmetry groups. This should not cause confusion.

[8]See, for example, R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); A. P. Balachandran, A. M. Din, J. S. Nilsson, and H. Rupertsberger, Phys. Rev. D 16, 1036 (1977); C. G. Callan, Jr., R. Dashen, and D. J. Gross, Institute for Advanced Study Report No. COO-2220-115, 1977 (to be published); H. Arfaei, to be published.

[9]P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. B125, 1 (1977).

[10]G. Mack, DESY Report No. 77/58, 1977 (to be published).

[11]W. Grueb, S. Halperin, and R. Vanstone, *Connections, Curvature and Cohomology II; Lie Groups, Principal Bundles and Characteristic Classes* (Academic, New York, 1973), p. 92.

[12]For, clearly, C is contained in L_Λ. Futhermore, C is a maximal commuting set in L and so in L_Λ. Hence the result.

[13]See for example, A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento 30, 845 (1963); J. Schechter, Y. Ueda, and S. Okubo, Ann. Phys. (N.Y.) 32, 424 (1965), and references therein.

[14]The effect of this bound on the quantum field theory has been considered by H. S. Sharatchandra, to be published.

[15]Here the generic elements belong to the general stratum in the sense of L. Michel and L. A. Radicati, Ann. Inst. Henri Poincaré 18, 185 (1973). The remaining elements are nongeneric.