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Anomalous Dimensionalities in the Spin-Glass Problem

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Arguments are given that the spin-glass problem has lower critical dimensionalities (below which no transition takes place) of 3 for vector spin-glasses and ≥ 2 for Ising spin-glasses. Thus for vector spin-glasses, the physical case is also at lower critical dimensionality, which is relevant to the peculiar behavior observed in classical spin-glass alloys.

The treatment by Harris, Lubensky, and Chen[1] of the Edwards-Anderson[2] replica theory suggests that the upper critical dimensionality of the spin-glass system (the dimensionality at which mean-field theory fails) is 6, not 4 as it is in most conventional phase transitions. Young[3] has, by real-space renormalization methods, suggested a lower critical dimensionality (lcd, below which order disappears) of around 2 for the Ising spin-glass, and one of the present authors has adapted the Harris-Lubensky-Chen treatment to suggest an lcd of 3 for the vector–spin-glass case.[4] In our opinion, the experimental data[5] on Heisenberg glasses in three dimensions strongly suggests that this system is at, or close to, lcd, and may even have a line of critical points ending at $T = 0$ like the 2D (two-dimensional) XY model and the 1D r^{-2} Ising model. (The evidence for this is the so-called "clustering" phenomena, clearly involving the weak dipolar anisotropy, which should not strongly affect an ordered system far from criticality and yet does so nonetheless over a wide range of temperature. A second point is that a magnetic field destroys the critical point as it does in a ferromagnet; yet at $H = 0$ there is no evidence of a true two-phase region —rather the line $H = 0$, $T < T_c$ seems more like a region of anomalous fluctuations.)

We try first to give strong plausibility arguments based on a version of real-space scaling that the lower critical dimensionality is 3 for the vector spin-glass.

First, let us define the order parameter we shall use for a vector spin-glass. While it is not certain, especially at low dimensionality, that a spin-glass has a unique ground-state configuration, Walker and Walstedt[6] have noted that, according to their computations, in local regions the spin arrangements remain reasonably rigid in different low-energy configurations. Different attempts at the ground state vary only in relative orientation of quite distant regions—such as one might expect with large long-range fluctuations as at lcd. Let us then assume a lowest-energy overall configuration locally, $\vec{S}_i^0$, and define a tensor order parameter in the manner of Binder and Schröder[7]: For a given configuration of average spins $\langle\vec{S}_i\rangle$,

$$\overleftrightarrow{q}_i = \langle\vec{S}_i\rangle\vec{S}_i^0. \tag{1}$$

We use $\langle\,\rangle$ to signify the thermal average, a bar to signify the average over sites i, and normalize so that

$$|\vec{S}_i{}^0|^2 = 1.$$

The magnitude of the order parameter is then given by

$$q^2 = \mathrm{Tr}\,\overleftrightarrow{q}\,\tilde{\overleftrightarrow{q}} = \overline{|\langle\vec{S}_i\rangle|^2}\;|\hat{S}_i{}^0|^2 = \overline{|\langle\vec{S}_i\rangle|^2}\,, \tag{2}$$

where $\tilde{\overleftrightarrow{q}}$ is the transpose of $\overleftrightarrow{q}$ and $\hat{S}_i{}^0$ are unit vectors.

Aside from this magnitude, in general $\langle\vec{S}_i\rangle$ will be rotated from $\vec{S}_i{}^0$ by an arbitrary angle, $\theta(\vec{r}_i)$, which is assumed to vary slowly with distance. A uniform rotation, $\theta(\vec{r}) = \theta_0$, independent of $\vec{r}$, will leave the energy unchanged, and we may characterize the local state by the rotations $\theta(\vec{r}_i)$ if the system is not at a minimum-energy state. Thus rigid rotations of an n-dimensional coordinate system may be the group of the order parameter[8] or at least may be contained in it (if, for instance, there is also a zero-point entropy). For $n = 2$, for instance, we have phase angle φ (a rotation in two dimensions) defined everywhere; for $n = 3$, a rotation vector $\vec{\theta}$ in some arbitrary direction, defining a projective sphere in 3-space (from $|\vec{\theta}| = 0$ to $|\vec{\theta}| = \pi$).

For discussing lcd behavior, it is only necessary to study quite low temperatures where, in general, φ or $\vec{\theta}$ will vary relatively slowly in space. We assume

$$Z = e^{-\beta\mathcal{H}}$$

with

$$\mathcal{H} = \sum_{ij} J_{ij}\vec{S}_i\cdot\vec{S}_j, \tag{3}$$

where J_{ij} is a random variable with the typical feature of "frustration[9]" which leads to spin-glass behavior: The lowest-energy $\vec{S}_i$ are random and in general many (of order $\frac{1}{2}$) of the terms of (3) are positive (unfavorable). The simplest case for our purposes is the choice $\langle J_{ij}\rangle = 0$.

Now, let us attempt to use a real-space renormalization procedure which is similar both to the techniques used by Thouless in the localization problem[10] and to the Migdal potential moving technique.[11] We anticipate that, in a state in which we have a slow fluctuation of $\vec{\theta}$ in space with

$$\vec{\theta} = \vec{\theta}(\vec{r}), \quad (\nabla\theta)/\theta \ll 1,$$

we may write the energy as a functional of $\nabla\theta$ so that

$$E = \int f(\nabla\theta)\, d^3r, \tag{4}$$

and *a priori* we would assume that, since f must be an even function of $\nabla\theta$, it will start out as

$$f(\nabla\theta) = A|\nabla\theta|^2 + O(|\nabla\theta|^4). \tag{5}$$

What we will try to show here is that the coefficient A in (5) renormalizes and acquires anomalous dimensions in this case—or perhaps that (5) is not correct at all.

Let us imagine dividing the spin-glass into blocks of side L, large enough compared with a that each has a definite internal spin structure and thus a defined $\vec{\theta}$, but are otherwise arbitrary. What we would like to know is the following: How does the energy depend upon $(\vec{\theta}_\alpha - \vec{\theta}_\beta)^2$, where α and β are neighboring blocks, as a function of L?

In the simple ferromagnetic case, it is easy to estimate this. If we assume that $\vec{\theta}$ varies linearly with distance, the energy of any given bond will be $\sim(\theta_\alpha - \theta_\beta)^2/L^2 \sim L^{-2}$. The total number of distorted bonds is L^d, so that the energy goes as L^{d-2}. Thus 2 is the lower critical dimensionality, since for $d < 2$ the coupling energy decreases indefinitely with the scale. We may also do this by a Migdal-type argument. The Migdal approximation may be characterized (see Kadanoff[11]) as the apparently very rough procedure of changing scale by simply moving bonds parallel to themselves to coincide with neighboring ones. It may be justified in the limits of very strong or very weak interactions—in the latter, by a linearity argument; in the former (which is the case relevant for lcd considerations), by the argument that since neighbors are nearly rigidly coupled, the bond between spins 1, 2 in a square of bonds and spins 1, 2, 3, 4 might as well be between 3, 4 since 2, 3 and 1, 4 are rigidly coupled. Having added enough bonds in parallel, one can reduce the problem to linear chains which have a characteristic series addition property:

$$\mathop{\mathrm{Tr}}_{\vec{S}_2} \exp[-\beta J_{12}(\vec{S}_1\cdot\vec{S}_2) - \beta J_{23}\vec{S}_2\cdot\vec{S}_3] \simeq \exp(-\beta J_{13}{}'\vec{S}_1\cdot\vec{S}_3),$$

where for the Ising case (strong-coupling limit)

$$J_{13}' = \mathrm{sgn}(J_{12}J_{23})\max(J_{12}, J_{23})$$

unless $|J_{12}| = |J_{23}|$, while in the vector case (strong-coupling limit)

$$\frac{1}{J_{13}'} = \mathrm{sgn}(J_{12}J_{23})\left(\frac{1}{|J_{12}|} + \frac{1}{|J_{23}|}\right)$$

since this is simply a case of adding elastic compliances and the direction of the intermediate spin can adjust itself at will.

In the Ising case, the Migdal scaling must be handled with care and is very dependent, as we see, on the scaling of the J distribution. Kirkpatrick[12] has attempted this case for the spin-glass by the Migdal method. In the vector case, however, the results are very simple at least to lowest order. For the ordinary ferromagnet, let z be the coordinate direction running from α to β. At a given value of z, all spins have the same configuration so that we might as well add up all the bonds, moving them to coincide. The sum is proportional to the area L^{d-1}. Then we take these bonds and note that they must be added in series to find the coupling energy since series bonds for vector spins add as inverses:

$$\frac{1}{J_{\mathrm{eff}}} = \sum \frac{1}{J} = \frac{L}{L^{d-1}J}, \quad J_{\mathrm{eff}} \sim L^{d-2}. \tag{6}$$

What this is saying is that this term in the free-energy functional essentially renormalizes only according to its naive dimensionality. As José *et al.*[13] point out, at $d = 2$ one must go to higher order in a series in J or $1/J$; specifically

$$\frac{d \ln J_{\mathrm{eff}}}{d \ln L} = d - 2 + \mathrm{const}\,\frac{n-2}{J_{\mathrm{eff}}} \tag{7}$$

and the constant is not quite accurately determined by the Migdal procedure. Estimates of such logarithmic terms are necessary to determine whether or not there is a phase transition at lcd.

The naive Migdal procedure gives quite a different answer for the spin-glass. Adding bonds in parallel, we have to note that they are random in sign; hence adding L^{d-1} bonds leads to a sum proportional to $L^{(d-1)/2}$ so that

$$J_{\parallel}^2 = L^{d-1}\,\overline{J_{ij}^2}\,. \tag{8}$$

On the other hand, since we allow the spins to assume their equilibrium value at each z, and only study rotation of the equilibrium value, these parallel bonds add in series exactly as they would if all ferromagnetic, according to.[7] Thus we arrive at the "naive" result

$$J_{\mathrm{eff}}(L) = L^{(d-1)/2-1}J_0 = L^{(d-3)/2}J_0. \tag{9}$$

This clearly gives an lcd of 3.

The straightforward Migdal approximation is much less clearly justified in this case, since it leads to a genuine renormalization not simply controlled by the dimensionality. We want to give some rather plausible arguments that, as in the ferromagnetic system, this result is valid near, and especially at, lcd. The argument may be put in a specific way. Imagine hypercubes of side L, at first wholly uncoupled. Each hypercube is presumed to be in or near its structure of minimum energy and its state is thus assumed to be defined by a rotation θ_α of this structure. If we now weakly couple two of these hypercubes by adding weak bonds at the surface (of area L^{d-1}) between them, the energies of these bonds are determined by the internal structure and can lead to a coupling between them only of strength $\propto L^{(d-1)/2}$. This coupling is arbitrary in sign, but, of course, for lowest energy the cubes will reorient their relative θ's to optimize it. So long as the extra bonds satisfy $J_{\mathrm{surf}} < L^{-1}J_{\mathrm{bulk}}$ a relative twist of the two cubes will be slightly responded to by a twist of the internal structure—but mostly at the surface; but if $J_{\mathrm{bulk}} \gg J_{\mathrm{surf}} \gg L^{-1}J_{\mathrm{bulk}}$, the response will be mostly in the bulk, although J_{surf} is still too weak to reorient the structure as a whole. (It is clear that the twisting energy and coupling energy are always proportional.) In fact, we can see that in this range the relaxation has reduced the twisting energy to $1/L$ of that if the twist took place purely at the surface.

There seems to be no reason to expect radical discontinuity in behavior as $J_{\mathrm{surf}} \sim J_{\mathrm{bulk}}$. Adding the surface bonds will in general change the structure around them, which may effectively weaken or strengthen them somewhat, but not change the balance of favorable or unfavorable ones. It is important not to confuse the total exchange energy with the order-parameter stiffness. Naturally every bond makes some small contribution to the total exchange energy, because it will perturb the structure around it by a finite amount. Thus the total exchange energy due to added surface bonds is $\sim L^{d-1}$. But we are interested in comparing the relative energy if we rotate one block's internal structure by $\theta = 180°$ relative to the others, and this difference is the difference between the number of surface bonds which are favorable to one structure and those which are unfavorable—which must be essentially the variance of the exchange coupling, $\sim L^{(d-1)/2}$. This argument seems to us quite unavoidable.

With this simple argument we cannot come to any conclusion as to whether a phase transition occurs at $d=3$. It is rather likely that, as for the normal ferromagnet at $d=2$, there is never any true long-range order, but for the $n=2$ XY model, that does not preclude a phase transition, which certainly occurs in that case.[13] It is possible that the observed phase transition in the spin-glass is real and intrinsic to Heisenberg spins, or that it is caused by small anisotropies which remove the $n=3$ character of the spins.

It is possible to produce an argument that the lcd for the Ising spin-glass is 2 or greater. The lcd for localization is 2, as is easily seen from Ref. 10. In an earlier paper[14] one of us showed that a phase transition in a spin-glass must imply that an extended eigenvalue of the magnetic response function χ_{ij} defined by

$$\langle M\rangle_i = \sum_j \chi_{ij} H_j$$

must diverge. χ_{ij} is a random matrix, and if it has no extended eigenfunctions no transition can occur. The very extensive simulations[7] of Binder and Schröder make it lekely that a transition does occur at $d=2$, but their results also suggest an absence of long-range order as might be expected; so the best conjecture is clearly an lcd of 2 (Ising). In conclusion, we note that from the theoretical phase/transition point of view, it is fascinating to have an experimentally accessible —and in fact much investigated—example of a system at lcd. It seems likely that many of the mysterious properties of the spin-glass systems are a result of this fact.

A second very intriguing question is whether this behavior is confined to the spin-glass, or does it also extend to other amorphous systems, and in particular to real glass or to random polymer systems of various kinds. The very simple argument leading to $(d-3)/2$ has no obvious limitation to the spin-glass case.

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