Johnson, T. G. Masterson, K. L. Erdman, A. W. Thomas, and R. H. Landau, to be published.

[4]S. A. Dytman, J. F. Amann, P. D. Barnes, J. N. Craig, K. G. R. Doss, R. A. Eisenstein, J. D. Sherman, W. R. Wharton, R. J. Peterson, G. R. Burleson, S. L. Verbeck, and H. A. Thiessen, Phys. Rev. Lett. 38, 1059 (1977), and 39, 53(E) (1977).

[5]J. Källne and A. W. Obst, Phys. Rev. C 15, 477 (1977).

[6]M. Dillig and M. G. Huber, Phys. Lett. 69B, 429 (1977).

[7]S. A. Dytman, to be published.

[8]B. M. Preedom, C. W. Darden, R. D. Edge, T. Marks, M. J. Saltmarsh, K. Gabathuler, E. E. Gross, C. A. Ludemann, P. Y. Bertin, M. Blecher, K. Gotow, J. Alster, R. L. Burman, J. P. Perroud, R. P. Redwine, B. Goplen, W. R. Gibbs, and E. L. Lomon, Phys. Lett. 65B, 31 (1976).

[9]F. Ajzenberg-Selove, Nucl. Phys. A248, 1 (1975).

[10]S. Dahlgren, P. Grafström, B. Hoistad, and A. Åsberg, Nucl. Phys. A211, 243 (1973).

[11]M. R. Meder and J. E. Purcell, Phys. Rev. C 12, 2056 (1975).

[12]G. J. Igo, T. Bauer, G. Pauletta, J. Soukup, C. A. Whitten, Jr., G. Blanpied, R. Liljestrand, G. W. Hoffmann, M. Oothoudt, and R. L. Boudrie, Bull. Am. Phys. Soc. 23, 47 (1978); G. Hoffmann, private communication.

[13]S. W. Cosper, R. L. McGrath, J. Cerny, C. C. Maples, G. W. Goth, and D. G. Fleming, Phys. Rev. 176, 1113 (1968).

[14]D. Bachelier, J. L. Boyard, T. Hennino, J. C. Lourdain, P. Radvanyi, and M. Roy-Stephan, Phys. Rev. C 15, 2139 (1977).

[15]R. A. Eisenstein and G. A. Miller, Phys. Rev. C 11, 2001 (1975); Michel Betz, thesis, Massachusetts Institute of Technology, 1977 (unpublished); A. K. Kerman, unpublished.

Theory of a Free-Electron Laser

Ira B. Bernstein and J. L. Hirshfield
Mason Laboratory, Yale University, New Haven, Connecticut 06520
(Received 18 August 1977)

A classical, linear theory is given for the gain of a short-wavelength free-electron laser. Waves propagating along the relativistic beam are unamplified when the beam is cold. Two of the six geometric optics modes have phase velocities $< c$. When thermal effects are included, the theory predicts the possibility of substantial amplification of these two modes for asymmetric electron distribution functions, due to wave-particle resonance.

Two recent experiments on the Stanford Linear Accelerator Center's superconductive linear accelerator have rekindled wide interest in the possibility of free-electron lasers.[1,2] A classical single-particle radiation analysis for this system was published in 1951 by Motz,[3] who later carried out experiments[4] with 100-MeV electrons. Phillips[5] posed the problem as an O-type traveling-wave interaction, and successfully built and operated microwave devices (Ubitrons) of high efficiency (~ 10%) and high power output (~ 800 kW). Quantum-mechanical calculations, posing the interaction as stimulated Compton scattering—a nonlinear process—have been published by Madey,[6] Sukhatme and Wolf,[7] and by Colson.[8] Hopf *et al.*[9] have reproduced Madey's result using a classical theory based on the relativistic collisionless Boltzmann equation, and have also published a strong-signal theory.[10] General features of stimulated processes leading to gain have been summarized by Granatstein and Sprangle.[11] Kwan, Dawson, and Lin[12] have carried out a linear stability analysis for a cold beam employing periodic boundary conditions and compared the predicted temporal growth rates with numerical simulations.

The linearized theory presented here invokes a mechanism not included in the above, namely wave-particle resonance, but neglects the free-streaming effects which underlie prior work. It indicates that within the geometric-optics approximation, a cold relativistic beam propagating along the axis of a helical static magnetic field does not amplify short-wavelength radiation propagating along the beam. Moreover, it suggests that sizable gain can be obtained using a beam with a finite momentum spread, but that this spread must be asymmetric. When the gain mechanism described here is operative it is likely that the above nonlinear theories must be modified so as to incorporate it.

We take the periodic transverse static magnetic field to be the curl of a vector potential

$$\vec{A}(z) = (mc^2/e)\vec{V}(z), \tag{1}$$

where $\vec{V}(z) = \vec{e}_x V_x(z) + \vec{e}_y V_y(z)$ is the dimensionless wiggle momentum with $V_x(z)$ and $V_y(z)$ periodic functions of z. This vector potential is valid near the axis of a system of oblate helically wound wires carrying zero net current. The choice $V_x = V\cos k_0 z$ and $V_y = V\sin k_0 z$ corresponds to the experiments[1,2]; the case $V_y = 0$ can be realized, for example, by an array of permanent magnets. As will be seen, the results do not depend critically upon the relative size of V_x and V_y.

The relativistic Lagrangian for an electron in the field given by (1), neglecting self-electrostatic effects, yields three constants of the motion:

$$\alpha = \gamma\dot{x}/c - V_x; \quad \beta = \gamma\dot{y}/c - V_y;$$

$$\gamma = [1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2]^{-1/2}. \tag{2}$$

Thus $f = f_0(\alpha, \beta, \gamma)$ is the corresponding general steady-state distribution function, without limit on the wiggle momentum.

The linearized kinetic equation for f_1, ths small perturbation in the distribution function, is

$$\partial f_1/\partial t + (c\vec{u}/\gamma)\cdot\nabla f_1 - (e/mc)(\vec{E}_1 + \vec{u}\times\vec{B}_1/\gamma)\cdot\nabla_u f_0 - (e/m\gamma)\vec{u}\times[\nabla\times(m\vec{V}c^2/e)]\cdot\nabla_u f_1 = 0, \tag{3}$$

where $\vec{u}$ is the electron momentum divided by mc, and $\vec{E}_1$ and $\vec{B}_1$ are the perturbed electromagnetic fields. In terms of the independent variables $\vec{r}$, t, α, β, and $u = (\gamma^2 - 1)^{1/2}$, (3) becomes

$$\partial f_1/\partial t + (c\vec{u}/\gamma)\cdot\nabla f_1 = (e/mc)(\vec{E}_1 + \vec{u}\times\vec{B}_1/\gamma)\cdot[(\vec{u}/u)\partial f_0/\partial u + \vec{e}_y\,\partial f_0/\partial\beta + \vec{e}_x\,\partial f_0/\partial\alpha]. \tag{4}$$

Since we are interested in interactions with electromagnetic wavelengths much shorter than the periodicity length in $\vec{V}(z)$, and propagation along the beam, we seek a steady-state geometric-optics[13] solution with $\vec{E}_1 = \vec{a}\exp[i(\Psi - \omega t)]$ and $\vec{B}_1 = (c\vec{k}\times\vec{a}/\omega)\exp[i(\Psi - \omega t)]$, where $\vec{k} = k(z)\vec{e}_z$ and $\Psi = \int dz\,k(z)$. In addition one has $f_1 = g(z)\exp[i(\Psi - \omega t)]$, and current density $\vec{J}_1 = \vec{j}(z)\exp[i(\Psi - \omega t)]$, with $\vec{j}(z) = -ec\int d^3u\,\vec{u}g/\gamma$. The quantities $\vec{a}(z)$, $k(z)$, $g(z)$, and $\vec{j}(z)$ are all slowly varying in a wavelength $2\pi/k$.

We solve for the conductivity tensor $\overleftrightarrow{\sigma}$ for the electron gas, where $\vec{j} = \overleftrightarrow{\sigma}\cdot\vec{a}$, and find

$$\overleftrightarrow{\sigma} = -\frac{ie^2}{m\omega}\int d\alpha\,d\beta\,du\,f_0(\alpha, \beta, u)\left\{\frac{\partial}{\partial u}\left[\frac{\vec{u}\vec{u}}{u_z^2}\,\frac{\omega}{kc - \omega\gamma/u_z}\right] - \frac{\partial}{\partial\beta}\left[\frac{u\vec{u}\vec{e}_y}{\gamma u_z}\right] - \frac{\partial}{\partial\alpha}\left[\frac{u\vec{u}\vec{e}_x}{\gamma u_z}\right]\right\}. \tag{5}$$

First we consider a cold beam; e.g., $f_0(\alpha, \beta, u) = N_0\delta(\alpha)\delta(\beta)\delta(u - v)$. Then (5) becomes ($\omega_p^2 = 4\pi N_0 e^2/m$)

$$\overleftrightarrow{\sigma} = \frac{i\omega_p^2}{4\pi\omega}\left\{\frac{v}{\gamma v_z^3}[u_z^2(\overleftrightarrow{I} - \vec{e}_z\vec{e}_z) + \vec{V}\vec{V}] + \frac{v}{v_z}\,\frac{\vec{V}(\vec{V} + \vec{e}_z v_z) + (\vec{V} + \vec{e}_z v_z)\vec{V}}{kc/\omega - \gamma/v_z} + \frac{v(1 + V^2)}{\gamma v_z^3}\,\frac{(\vec{V} + \vec{e}_z v_z)(\vec{V} + \vec{e}_z v_z)}{(kc/\omega - \gamma/v_z)^2}\right\}, \tag{6}$$

where now $v_z^2 = v^2 - V^2$ and $\gamma = (1 + v^2)^{1/2}$. The dielectric tensor $\overleftrightarrow{\epsilon} = (1 - k^2c^2/\omega^2)\overleftrightarrow{I} + c^2\vec{k}\vec{k}/\omega^2 + 4\pi i\overleftrightarrow{\sigma}/\omega$ may now be formed. The normal modes of the system obey $\overleftrightarrow{\epsilon}\cdot\vec{a} = 0$. It is convenient to use the right-handed orthonormal basis vectors $\vec{e}_1 = \vec{e}_z\times\vec{V}/V$, $\vec{e}_2 = \vec{V}/V$, and $\vec{e}_3 = \vec{e}_z$. Then

$$\overleftrightarrow{\epsilon} = \epsilon_{11}\vec{e}_1\vec{e}_1 + \epsilon_{22}\vec{e}_2\vec{e}_2 + \epsilon_{23}(\vec{e}_2\vec{e}_3 + \vec{e}_3\vec{e}_2) + \epsilon_{33}\vec{e}_3\vec{e}_3, \tag{7}$$

where

$$\begin{aligned}
\epsilon_{11} &= 1 - (kc/\omega)^2 - (\omega_p^2/\omega^2)(v/\gamma v_z),\\
\epsilon_{22} &= 1 - (kc/\omega)^2 - (\omega_p^2/\omega^2)(v/\gamma v_z^3)[v^2 + 2\gamma V^2/v_z\lambda + V^2(1 + V^2)/\lambda^2 v_z^2],\\
\epsilon_{23} &= -(\omega_p^2/\omega^2)(v/\gamma v_z^3)[\gamma V/\lambda + (1 + V^2)V/\lambda^2 v_z],\\
\epsilon_{33} &= 1 - (\omega_p^2/\omega^2)(v/\gamma v_z^3)(1 + V^2)/\lambda^2,
\end{aligned} \tag{8}$$

with $\lambda = kc/\omega - \gamma/v_z$. From $\det\overleftrightarrow{\epsilon} = 0$ we see that the roots of the six geometric optics modes are given by

$$\epsilon_{11} = 0 \text{ or } kc/\omega = \pm|1 - (\omega_p^2/\omega^2)(v/\gamma v_z)|^{1/2} \tag{9}$$

(waves with $\omega/k > c$), and by $D \equiv \epsilon_{22}\epsilon_{33} - \epsilon_{23}^2 = 0$, or, after some algebra,

$$0 = [1 - (\omega_p^2/\omega^2)(v/\gamma v_z) - (k^2c^2/\omega^2)][(kc/\omega - \gamma/v_z)^2 - (\omega_p^2/\omega^2)v/\gamma v_z^3]. \tag{10}$$

One pair of roots is identical to (9). The other pair is

$$kc/\omega=\gamma/v_z \pm (\omega_p/\omega)(v/\gamma v_z^3)^{1/2} \tag{11}$$

which are slow waves ($\omega/k < c$). The forms of (9) and (11) are reminiscent of corresponding forms for conventional transverse electromagnetic and longitudinal space-charge waves on a cold plasma stream, but it must be noted that here the modes are neither purely transverse ($\vec{k}\cdot\vec{a}=0$) nor purely longitudinal ($\vec{k}\times\vec{a}=0$). In any case, it is apparent that (9) and (11) lead to purely real roots: The cold beam manifestly does not amplify.

We now consider the effect of a small thermal spread in the beam momentum. The most important change will be the addition of a small real component to (6), on account of wave-particle coupling. The two modes given by (11) are thus the only candidates for growth. The possibility of amplification is governed by the steady-state transport equation for the wave energy U,[13]

$$\nabla\cdot(\omega_{\vec{k}}U)+\overleftrightarrow{\sigma}^{\mathrm{H}}:\vec{a}\vec{a}^*=0, \tag{12}$$

where $U=(\omega/8\pi)(\partial\overleftrightarrow{\epsilon}/\partial\omega):\vec{a}\vec{a}$, and $\overleftrightarrow{\sigma}^{\mathrm{H}}$ is the Hermitian part of $\overleftrightarrow{\sigma}$.

After some manipulation, one can show that when $v^2 \gg V^2 \simeq 1$,

$$U=\mp\frac{1}{4\pi}\frac{\omega}{\omega_p}\frac{1+V^2}{V^2}\frac{a^2}{v^{1/2}}, \tag{13}$$

where the upper sign (negative energy) goes with the analog of the slow space-charge wave given by the upper sign in (11), and where the lower sign (positive energy) goes with the analog of the fast space-charge wave given by the lower sign in (11). The group velocity can be shown to be $\partial\omega/\partial k=cv_z/\gamma\simeq c$, namely the z velocity of the beam. The lowest-order thermal contribution to $\overleftrightarrow{\sigma}^{\mathrm{H}}$ is, from (5),

$$\overleftrightarrow{\sigma}^{\mathrm{H}}=-\frac{\pi e^2}{m\omega}\int d\alpha\, d\beta\, du\,\frac{\partial f_0}{\partial u}\frac{\vec{u}\vec{u}}{u_z^2}\delta\left(\frac{kc}{\omega}-\frac{\gamma}{u_z}\right). \tag{14}$$

It can be shown that $\vec{u}\vec{u}:\vec{a}\vec{a}^*\simeq a^2/V^2$. Thus if we define $N_0F(u)=\int d\alpha\, d\beta f_0(\alpha,\beta,u)$, we have from (12)

$$\frac{\partial}{\partial z}\left[\mp\frac{c}{4\pi}\frac{\omega}{\omega_p}\frac{1+V^2}{V^2}\frac{a^2}{v^{1/2}}\right]-\frac{\omega_p^2}{4\omega}\frac{a^2v}{(1+V^2)V^2}F'(w_\pm)=0, \tag{15}$$

where $w_\pm\simeq v\mp(\omega_p/\omega)v^{3/2}(1+V^2)^{-1}$ and the upper and lower signs correspond to those in (11). For the case $\vec{V}=V(\vec{e}_x\cos k_0z+\vec{e}_y\sin k_0z)$, Eq. (15) yields

$$\frac{\partial}{\partial z}(\ln a^2)=\mp\pi\frac{\omega_p}{c}\left(\frac{\omega_p}{\omega}\right)^2\frac{v^{3/2}}{(1+V^2)^2}F'(w_\pm)\equiv G, \tag{16}$$

and growth occurs if $F'(w_+)<0$ for the slow space-charge wave, and if $F'(w_-)>0$ for the fast space-charge wave.

The consequences of (16) can be readily understood with the aid of Fig. 1. Clearly the symmetric distribution 1(a) leads to spatial decay since $F'(w_+)>0$ and $F'(w_-)<0$. But the asymmetric distribution 1(b), which is qualitatively like that of many relativistic beams, can yield spatial amplification for the positive-energy wave for $v<w_-<u_0$. If we approximate $F'(w_+)\simeq 1/\Delta^2$, where Δ is an effective width, then with $\omega=10^{14}$, $\omega_p=3\times10^9$, $v\simeq\gamma=50$, $V^2=0.5$, and $\Delta=10^{-2}$, (16) yields $G=10^{-4}\ \mathrm{cm}^{-1}=1\%/\mathrm{m}$.[1] A distribution function skewed in the opposite sense would amplify the negative-energy wave.

In conclusion, the gain of a free-electron laser described by the theory presented here depends critically on the axial momentum distribution of the beam. Asymmetry is required for gain, a low-energy tail leading to amplification of the positive-energy wave. Suitable tailoring of the distribution function could result in gains higher than those that have been observed. Clearly the results are very sensitive to the shape of $F(u)$ suggesting that appropriate experiments would be most valuable in establishing the predictions of this work.

We have benefitted from discussions with

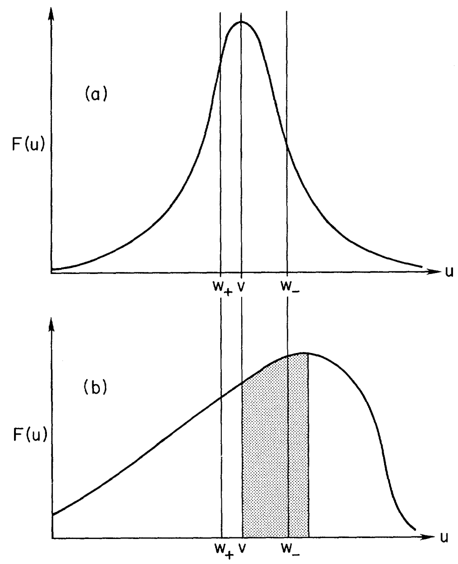

FIG. 1. Axial momentum distributions. In (a) a symmetric distribution is shown which is stable against spatial amplification for both the fast and slow space-charge waves, since $F'(w_+) > 0$ and $F'(w_-) < 0$. In (b) an asymmetric distribution similar to that furnished by a linac is shown which can support amplification for the fast wave, when w_- is in the shaded region where $F'(w_-) > 0$.

T. Godlove, V. Granatstein, K. R. Chu, R. Gullikson, and F. Firk. This work was partially supported by the National Science Foundation.

[1]L. R. Elias *et al.*, Phys. Rev. Lett. 36, 717 (1976).
[2]D. A. G. Deacon *et al.*, Phys. Rev. Lett. 38, 892 (1977).
[3]H. Motz, J. Appl. Phys. 22, 527 (1951).
[4]H. Motz *et al.*, J. Appl. Phys. 24, 826 (1953).
[5]R. M. Phillips, IRE Trans. Electron Devices 3, 231 (1960).
[6]J. M. J. Madey, J. Appl. Phys. 42, 1906 (1971).
[7]V. P. Sukhatme and P. A. Wolff, J. Appl. Phys. 44, 2331 (1973).
[8]W. B.Colson, Phys. Lett. 59A, 187 (1976).
[9]F. A. Hopf *et al.*, Opt. Commun. 18, 413 (1976).
[10]F. A. Hopf *et al.*, Phys. Rev. Lett. 37, 1342 (1976).
[11]V. L. Granatstein and P. Sprangle, IEEE Trans. Microwave Theory Tech. 25, 545 (1977).
[12]T. Kwan, J. M. Dawson, and A. T. Lin, Phys. Fluids 20, 581 (1977).
[13]Ira B. Bernstein, Phys. Fluids 18, 320 (1975).

Trapping of Cusp-Injected, Nonneutral, Electron Rings with Resistive Walls and Static Mirror Coils

D. W. Hudgings, R. A. Meger, C. D. Striffler, W. W. Destler, H. Kim, M. Reiser, and M. J. Rhee
Electrical Engineering Department and Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 23 November 1977)

A hollow, rotating 2.3-MeV electron beam, moving with axial velocity of ~0.25c after injection through a magnetic-field cusp, is slowed down and trapped in a small magnetic mirror well by interaction with a 30–60-Ω/square resistive wall. The trapped-electron ring containing ~10^{12} particles performs a damped oscillation about the well minimum and settles down into an equilibrium state.

The electron-ring accelerator (ERA) concept, first proposed by Veksler *et al.*,[1] uses the electrostatic potential well of a high-density, relativistic electron ring for collective acceleration of positive ions. Experiments in progress at several laboratories[2-4] employ a pulsed magnetic mirror field with azimuthal beam injection to obtain a compressed electron ring. In the ERA experiment at the University of Maryland,[5-6] on the other hand, the ring is being formed with the aid of a static, cusped magnetic field. The cusp transforms a long hollow electron beam into a short rotating cylindrical electron layer (E layer) which propagates in the axial direction at a fraction of the speed of light. The length and velocity of the E layer depend on the pulse shape of the injected electron beam and the axial magnetic field, B. As B is increased, the axial velocity is reduced until a cutoff value, B_c, is reached where all particles are reflected in the cusp region. Experimentally, it is found that reproducible beams are obtained only when B is some-