# Quantized Electric Flux Tubes in Quantum Chromodynamics

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It is shown that in the strong-coupling limit the spectrum of quantum chromodynamics consists of closed and open strings with quarks at the ends. A Hamiltonian formulation in the axial gauge $A_3^a = 0$ is used. The theory is reformulated in terms of new canonical variables. Algebraic methods are employed to arrive at these results.

I analyze the familiar SU($N$) quantum chromodynamics (QCD) in four dimensions. A gauge-invariant Hamiltonian density follows from the symmetric energy-momentum tensor $\theta_{\mu\nu}$, $\mathcal{H}=\theta_{00}$ up to constraints. The definition of the theory is completed by imposing the gauge-invariant boundary conditions $F_{\mu\nu}{}^a \to 0$, $\Psi_\alpha{}^i \to 0$ at space infinity $|\vec{x}|\to\infty$. In the quantum theory this condition defines the physical states, and in particular the vacuum, as we will see below.

I specialize to the axial gauge $A_3{}^a=0$. The canonical formulation, with particular attention to boundary terms, has been reanalyzed recently in two dimensions[1] and four dimensions.[2] In the axial gauge the canonical variables are $(A_i{}^a, E_i{}^a)$ $(i=1,2)$ and $(\psi, \psi^\dagger)$ while $A_0{}^a$ and $E_3{}^a = -\partial_3 A_0{}^a$ are dependent operators. From the constraint equation $D^\mu F_{0\mu}{}^a = gJ_0{}^a$ we solve for $E_3{}^a$:

$$g^{-1}E_3{}^a(x_\perp, z, t) = \int_z^\infty dz'\, G_3{}^a(x_\perp, z', t), \tag{1}$$

where $G_3{}^a = g^{-1}D_1E_1{}^a + g^{-1}D_2E_2{}^a + \psi^\dagger \frac{1}{2}\lambda^a\psi$.

$E_3{}^a(x_\perp, z=-\infty)$ is an operator with nontrivial commutation rules with the canonical variables. Therefore, imposing the boundary condition $E_3{}^a(x_\perp,-\infty)=0$ implies a constraint. Rather than solving this constraint[3] I choose to work in a larger Hilbert space and separate the physical sector by the condition[2,4,5] $\int_{-\infty}^{\infty} dz\, G_3{}^a|\text{phys}\rangle = 0$.

To analyze the strong-coupling limit I perform the canonical transformation[6] $\tilde{A}_i{}^a = gA_i{}^a$, $\tilde{E}_i{}^a = g^{-1}E_i{}^a$. The Hamiltonian density takes the form

$$\mathcal{H} = \tfrac{1}{2}g^2\tilde{E}^2 + \tfrac{1}{2}g^{-2}[(\partial_3 A_i)^2 + \tilde{F}_{12}{}^2] + \psi^\dagger(-\tfrac{1}{2}i\vec{\alpha}\cdot\overleftrightarrow{\nabla} + \beta m + \alpha_i \tilde{A}_i \cdot \tfrac{1}{2}\lambda)\psi.$$

Therefore, in the limit $g\to\infty$ the first term dominates $\mathcal{H}_0 = \frac{1}{2}g^2(\tilde{E}_1{}^2 + \tilde{E}_2{}^2 + \tilde{E}_3{}^2)$. The canonical rules lead to the algebra

$$[G_3{}^a(\vec{x}), G_3{}^b(\vec{x}')] = if^{abc}G_3{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}'), \tag{2}$$

$$[\tilde{E}_3{}^a(\vec{x}), \tilde{E}_3{}^b(\vec{x}')] = if^{abc}\delta^{(2)}(x_\perp - x_\perp')\{\theta(z-z')\tilde{E}_3{}^c(\vec{x}) + \theta(z'-z)\tilde{E}_3{}^c(\vec{x}')\}, \tag{3}$$

$$[\tilde{E}_3{}^a(\vec{x}), \tilde{E}_i{}^b(\vec{x}')] = if^{abc}\delta^{(2)}(x_\perp - x_\perp')\theta(z'-z)\tilde{E}_i{}^c(\vec{x}'). \tag{4}$$

From this algebra it is straightforward to derive that $\tilde{E}_1{}^2$, $\tilde{E}_2{}^2$, $\tilde{E}_3{}^2$, $G_3{}^2$ *commute with each other at all points in space.* Therefore, $H_0 = \int d^3x\,\mathcal{H}_0$ consists of commuting pieces which can be diagonalized at each space point independently. Hence, the eigenstates of $H_0$ can be written in the form of a direct product $\prod_n |e_1{}^2(\vec{x}_n), e_2{}^2(\vec{x}_n), e_3{}^2(\vec{x}_n); g_3{}^2(\vec{x}_n), \dots\rangle$, where the dimensionless numbers $(e_I{}^2, g_3{}^2, \dots)$ are, up to a scale, the eigenvalues of the operators $(\tilde{E}_I{}^2, G_3{}^2, \dots)$ at the point $\vec{x}_n$. I expect a larger complete set of commuting operators as will be shown below. The task is to impose the canonical commutation rules on the states in order to discover relations among the complete set of labels $(e_1{}^2, e_2{}^2, e_3{}^2, g_3{}^2, \dots)$. For this purpose I find it extremely useful and illuminating to introduce new variables.

I introduce $\Pi_1 = \frac{1}{2}\lambda^a\Pi_1{}^a$, $\Pi_2 = \frac{1}{2}\lambda^a\Pi_2{}^a$, and the $N\times N$ unitary matrices $B_{13}{}^{ij}$, $B_{23}{}^{ij}$ $(B_{13}{}^\dagger B_{13} = B_{23}{}^\dagger B_{23})$. I rewrite the canonical variables $(\tilde{A}_i{}^a, \tilde{E}_i{}^a)$ as

$$\tfrac{1}{2}\lambda^a\tilde{A}_1{}^a = iB_{13}{}^\dagger\partial_1 B_{13}, \quad \tfrac{1}{2}\lambda^a\tilde{A}_2{}^a = iB_{23}{}^\dagger\partial_2 B_{23}, \quad \tfrac{1}{2}\lambda^a\tilde{E}_1{}^a = :B_{13}{}^\dagger\Pi_1 B_{13}:, \quad \tfrac{1}{2}\lambda^a\tilde{E}_2{}^a = :B_{23}{}^\dagger\Pi_2 B_{23}:, \tag{5}$$

where the notation indicates an operator ordering, $:B^\dagger\Pi B: \equiv B^\dagger(\lambda^a/2)B\Pi^a = \Pi^a B^\dagger(\lambda^a/2)B = B^\dagger\Pi B + \frac{1}{2}(N - N^{-1})\theta(0)\delta^{(2)}(0)$, in accordance with the tracelessness of $\lambda\cdot\tilde{E}$ and the commutation rules specified below. Then, the canonical rules $[\tilde{A}_1{}^a, \tilde{A}_1{}^b] = 0 = [\tilde{E}_1{}^a, \tilde{E}_1{}^b]$, $[\tilde{A}_1{}^a, \tilde{E}_1{}^b] = i\delta^{(3)}(\vec{x}-\vec{x}')\delta^{ab}$, etc., are reproduced as follows: (a) All $B_{13}{}^{ij}, B_{23}{}^{ij}, (B_{13}{}^\dagger)^{ij}, (B_{23}{}^\dagger)^{ij}$ commute with each other. (b) All variables with label "1" commute with all variables with label "2." (c) $B$ and $\Pi$ commute with the quark variables $\psi$. (d) The nontrivial commutators are

$$[\Pi_1{}^a(\vec{x}), B_{13}{}^{ij}(\vec{x}')] = -[\tfrac{1}{2}\lambda^a B_{13}(\vec{x}')]^{ij}\theta(x_1'-x_1)\delta(x_2'-x_2)\delta(x_3'-x_3). \tag{6}$$

The Hermitian conjugate of (6) gives $[\Pi^a, (B^\dagger)^{ij}]$. Using Jacobi identifies one obtains

$$[\Pi_1{}^a(\vec{x}), \Pi_1{}^b(\vec{x}')] = if^{abc}\delta(x_2-x_2')\delta(x_3-x_3')\{\theta(x_1-x_1')\Pi_1{}^c(\vec{x}) + \theta(x_1'-x_1)\Pi_1{}^c(\vec{x}')\}. \tag{7}$$

Similar equations hold for the variables $(B_{23}, \Pi_2)$ for which $\theta(x_2'-x_2)$ appears. Note the identity $\frac{1}{2}\lambda_{ij}{}^a\lambda_{kl}{}^a = \frac{1}{2}(B^\dagger\lambda^a B)_{ij}(B^\dagger\lambda^a B)_{kl} = \delta_{il}\delta_{kj} - N^{-1}\delta_{ij}\delta_{kl}$.

With the new variables I obtain $D_1\tilde{E}_1 = :B_{13}{}^\dagger\partial_1\Pi_1 B_{13}:$, $D_2\tilde{E}_2 = :B_{23}{}^\dagger\partial_2\Pi_2 B_{23}:$, so that $G_3{}^a$ and $E_3{}^a$ in Eq. (1) can be written in terms of $B$ and $\Pi$. Thus, from now on I use (6) and (7) as the basic commutators.

The commutation rules, Eq. (7), of $\Pi_1{}^a$ (and $\Pi_2{}^a$) are analogous to those of $\tilde{E}_3{}^a$ [Eq. (3)]. I will denote $\tilde{E}_3{}^a = \Pi_3{}^a$ from here on. Furthermore I define $G_1{}^a$, $G_2{}^a$ satisfying relations similar to Eq. (1) $G_1{}^a = -\partial_1\Pi_1{}^a$, $G_2{}^a = -\partial_2\Pi_2{}^a$, $G_3{}^a = -\partial_3\Pi_3{}^a$. Using Eq. (7) I derive the remarkable algebra

$$[G_I{}^a(\vec{x}), G_J{}^b(\vec{x}')] = if^{abc}G_I{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\delta_{IJ}, \tag{8}$$

where $I, J = 1, 2, 3$, correspond to three local commuting SU($N$) groups.

Further insight can be gained by noticing that $B_{13}$ (or $B_{23}$) is the *operator* gauge transformation from the axial gauge $A_3=0$ to $A_1=0$ (or $A_2=0$). Then, $\Pi_1$ (or $\Pi_2$) is the analog of $\tilde{E}_3$ in the appropriate gauge $A_1=0$ (or $A_2=0$). I define $B_{12}=B_{13}B_{23}{}^\dagger$ which maps the axial gauge $A_2=0$ into $A_1=0$. Then, using Eq. (6) I derive that $G_I{}^a$ act as the generators of chirallike gauge transformations on $B_{12}, B_{13}, B_{23}$:

$$[G_I{}^a, B_{JK}{}^{ij}] = \{\delta_{IJ}(\tfrac{1}{2}\lambda^a B_{JK})^{ij} - \delta_{IK}(B_{JK}\tfrac{1}{2}\lambda^a)^{ij}\}\delta^{(3)}(\vec{x}-\vec{x}'). \tag{9}$$

The structure of the three local commuting SU($N$) groups is now quite transparent. It is convenient to use the notation $B_{IJ}=B_{JI}{}^\dagger$ ($B_{IJ}=1$, if $I=J$). The constraint following Eq. (1) takes the form

$$\lambda^a G_3{}^a + :B_{31}G_1{}^a\lambda^a B_{13}: + :B_{32}G_2{}^a\lambda^a B_{23}: = \rho_3. \tag{10}$$

The quark color density $\rho_3{}^{ij}$ (which also satisfies a closed algebra) can be written in three equivalent ways which exhibit transformations into the gauges $A_1=0$ and $A_2=0$:

$$\rho_3{}^{ij} = \psi^\dagger\tfrac{1}{2}\lambda^a\psi(\lambda^a)^{ij} = \psi^\dagger B_{31}\tfrac{1}{2}\lambda^a B_{13}\psi(B_{31}\lambda^a B_{13})^{ij} = \psi^\dagger B_{32}\tfrac{1}{2}\lambda^a B_{23}\psi(B_{32}\lambda^a B_{23})^{ij}. \tag{11}$$

I now interpret $\Pi_I{}^a$ as sums of generators (like sum of angular momenta) along straight lines embedded in three dimensions, since I can write

$$\Pi_1{}^a = \int_{x_1}^\infty dx_1' G_1{}^a, \quad \Pi_2{}^a = \int_{x_2}^\infty dx_2' G_2{}^a, \quad \Pi_3{}^a = \int_{x_3}^\infty dx_3' G_3{}^a. \tag{12}$$

The *boundary conditions* to be applied on physical states are

$$\Pi_1{}^a(-\infty, x_2, x_3) = \Pi_2{}^a(x_1, -\infty, x_3) = \Pi_3{}^a(x_1, x_2, -\infty) = 0. \tag{13}$$

Equations (8) and (12) give $[\Pi_I{}^a, G_J{}^2]=0$ which shows that we can diagonalize simultaneously the larger set of operators $G_1{}^2, G_2{}^2, G_3{}^2, \Pi_1{}^2, \Pi_2{}^2, \Pi_3{}^2$ at each space-time point $\vec{x}$ independently. Furthermore Eq. (5) allows us to write ($E_1{}^2=\Pi_1{}^2$) $\mathcal{H}_0 = \frac{1}{2}g^2(\Pi_1{}^2+\Pi_2{}^2+\Pi_3{}^2)$ which has eigenstates of the form $\Pi|e_I{}^2(x_n), g_I{}^2(x_n), \ldots\rangle$, where the dimensionless $(e_I{}^2, g_I{}^2, \ldots)$ are the eigenvalues of $(\Pi_I{}^2, G_I{}^2, \ldots)$ up to a scale.

The commutation relations which I have just worked out amount to these rules: (1) We should expect that $g_I{}^2(x_n)$ are determined by quantized pure numbers since $G_I{}^2(x_n)$ are Casimir operators of three commuting non-Abelian Lie algebras at each $x$ [like $J^2=j(j+1)$ for angular momentum]. (2) Since $\Pi_I{}^a$ are sums of generators [Eq. (12)] (like total angular momentum) $\Pi_I{}^2$ must be determined by quantized numbers as well. (3) Furthermore, given a state with a configuration of points $(\vec{x}_n)$ at which $g_I^2(\vec{x}_n) \neq 0$ are specified, then the Clebsch-Gordan series puts a restriction, in terms of $g_I^2(\vec{x}_n)$, on the maximum and minimum quantum numbers allowed for $e_I^2(x_n)$ (like $|j_1 - j_2| \leq j \leq j_1 + j_2$ for the sum of angular momenta). (4) The boundary conditions Eq. (13) mean that the total quantum number corresponding to the sum of generators along any line from $-\infty$ to $+\infty$ must be 0 on physical states. (5) The constraint Eq. (10) puts a restriction among $(g_1^2, g_2^2, g_3^2)$ at a given point $\vec{x}_n$, such that they should add up to the quantum number specified by the quark color density $\rho^2(\vec{x}_n)$ (like adding three angular momenta).

Let us now be more specific on the eigenvalues of $G_I^2$, say $G_1^2$. Mathematically, this is ill defined since there is a $\delta^{(3)}(\vec{x} - \vec{x}')$ in the algebra (8) satisfied by the distribution $G_1^a$. I will introduce a *short distance* cutoff $\Delta$ to define the singular operator product $G_1^2(x)$ (and $\Pi_1^2$, etc.). This can be done by smoothing the $\delta$ function $\delta(x_1) \to (\Delta/\pi)(x_1^2 + \Delta^2)^{-1}$ etc., so that at the origin $\delta^{(3)}(0) = (\pi\Delta)^{-3}$. (An alternative approach is to introduce a lattice with a spacing of order $\pi\Delta$.) I will keep $\Delta$ finite from now on since I expect that the full theory must determine a scale parameter self-consistently, as indicated, e.g., by the renormalization group. In this Letter $\Delta$ will determine the thickness of the string. Eventually I expect it to be related to the Regge slope and of course to the first massive state.

I can now write $[G_1^a(x), G_1^b(x)] = (\pi\Delta)^{-3}\, i f^{abc} \times G_1^c(x)$ which shows that the dimensionless operator $(\pi\Delta)^3 G_1^a(x) = g_1^a(x)$ leads to pure quantum numbers determined by the Lie algebra SU($N$). For example, the Casimir operator is $g_1^2(x) = \frac{1}{2}(N - N^{-1})$ for the fundamental representation, and other quantum numbers for higher representations. One may, of course, have different quantum numbers at different points $\vec{x}_n$. With similar arguments I conclude that $(\pi\Delta)^2 \Pi_1^a(x) = e_1^a(x)$ leads to pure quantum numbers for $e_1^2(x)$. Thus, the electric field $\Pi_1^2 = \tilde{E}_1^2 = (\pi\Delta)^{-4} e_1^2$ is quantized, and so are $\tilde{E}_2^2$ and $\tilde{E}_3^2$.

The procedure for constructing some explicit states can be explained simply if we first specialize to a pure SU(2) gauge theory, without fermions in two space and one time dimensions. Afterwards I reintroduce the quarks, SU($N$), and four dimensions.

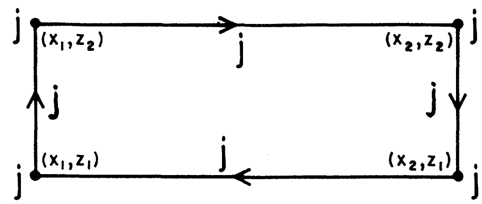


FIG. 1. Simplest closed string corresponding to Eq. (18).

Thus, eliminating the $y$ direction, note that the constraint (10) leads to $G_1^2 = G_3^2 = (\pi\Delta)^{-4} j(j+1)$ at each point $(x_n, z_n)$, where $j = 0, \frac{1}{2}, 1$, etc., for SU(2). Then, one sees the following:

(1) The state that has zero labels ($g^2 = e^2 = 0$) at all points $(x_n, z_n)$ is the vacuum with $H_0 = 0$. It satisfies

$$G_I^a(x)|0\rangle = 0 = \Pi_I^a(x)|0\rangle. \tag{14}$$

(2) There are no states with one, two, or three points with nonzero labels that satisfy the five rules of the previous paragraph.

(3) The simplest nontrivial state must have nonzero labels at four points which are at the corners of a rectangle as in Fig. 1. This arrangement is necessary to satisfy the boundary conditions (13), which also impose that $g_I^2(n) = j(j+1)$ must be identical at the four *corners*. By the rules of addition of angular momenta [Eq. (12)] I conclude that $e_1^2 = e_3^2 = j(j+1) =$ constant along the four *sides* of the rectangle, *and 0 everywhere else. Thus, this state corresponds to a closed string with quantized "electric" flux tubes* $[E_1^2 = E_3^2 = (\pi\Delta)^{-2} j(j+1)]$. The energy of the state is $H_0 = \frac{1}{2}g^2 \int dx\, dz (\Pi_1^2 + \Pi_3^2) = \frac{1}{2}g^2 L (\pi\Delta)^{-1} j(j+1)$, where $L$ is the length of the string (perimeter of rectangle) and $\pi\Delta$ is the thickness ($g^2$ has dimensions).

(4) By arranging an infinite number of points to satisfy the five rules, it is clear that the only states possible are closed strings of any shape.

(5) These states can be constructed explicitly by applying $B$ and $B^\dagger$ operators on the vacuum. For example the state corresponding to Fig. 1 with $j = \frac{1}{2}$ is

$$|S\rangle = \mathrm{Tr}[B_{31}(x_1 z_1) B_{13}(x_2 z_1) B_{31}(x_2 z_2) B_{13}(x_1 z_2)]|0\rangle.$$

We see that, e.g., $\Pi_1^2$ applied to this state gives

$$\Pi_1^2(x)|S\rangle = [\Pi_1^a(x), [\Pi_1^a(x), [\mathrm{Tr}\{\cdots\}]]]|0\rangle \approx |S\rangle (\pi\Delta)^{-1} \tfrac{1}{2}(\tfrac{1}{2}+1)\theta(x_2 - x)\theta(x - x_1)[\delta(z - z_1) + \delta(z - z_2)].$$

provided that the points are separated by distances larger than $\Delta$. Here I have used Eqs. (14), (12), and (9) with a smeared $\delta$ function. Similarly, $\Pi_3^2$ and $G_I^2$ are diagonal on $|S\rangle$. Thus this state is a closed string of quantized flux. The state for a string with arbitrary shape can be constructed by a straightforward generalization.

It is clear that the above discussion applies equally well in four dimensions and with any group SU($N$), leading to closed quantized flux tubes embedded in a three-dimensional space. When quarks are introduced we must watch the constraint (10), which makes it possible to have open strings. For example, it is now possible to have states on which $G_2{}^a = G_3{}^a = 0$, $G_1{}^a = \rho_1{}^a$, where $\rho_1{}^a = \Psi^+ B_{31} \frac{1}{2}\lambda^a B_{13}\Psi$ [Eq. (11)]. This leads to an open string with quarks at the ends lying along the x direction. More generally, *an open string with arbitrary shape would satisfy (10) by making two of the* $G_I{}^a = 0$ *at the ends, where the quarks are located* ($\rho \neq 0$), *while making* $\rho(x) = 0$ *and at least two nonzero* $G_I{}^a$ *at the intermediate points.* An explicit construction of such states is

$$\psi_\alpha{}^\dagger(\vec{x}_n) B_{3I}(\vec{x}_{n-1}) \cdots B_{JK}(\vec{x}_k) \cdots B_{L3}(\vec{x}_2)\psi_\beta(\vec{x}_1)|0\rangle.$$

For the SU(3) group $Y$-shaped baryons with three strings can be similarly constructed without difficulty.

The above are typical states with the *lowest*, constant energy density. Other states with higher, nonconstant, energy density can also be constructed via Clebsch-Gordan coefficients that couple the $(ij)$ indices on the $B$'s and $\psi$'s to higher representations.

Obviously there is some global similarity between the present results and the lattice gauge theory[6] of Wilson and Kogut and Suskind. However, the theories are different in detail. In particular I note that for $\Delta = 0$ my Hamiltonian, written in terms of either the old $(A, E)$ or the new $(\Pi, B)$ variables, is Poincaré covariant.[2] My cutoff method does not require any drastic changes in the form of the quantum theory. The $B$ operators are local, unlike the link operators on the lattice which are bilocal. Furthermore, at the location of the $B$ "electric" flux changes direction as in Fig. 1.

I have shown that, *in the strong-coupling limit*, QCD leads to closed strings as well as to open strings with quarks at the ends. To make further progress one must develop methods of calculation to include the neglected terms of the Hamiltonian. One must also obtain a better understanding of the deep question of the scale parameter $\Delta$, and investigate whether the cutoff can be removed by renormalization. These will be the goals of my future research.

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[1] I. Bars and H. B. Green, Phys. Rev. D 17, 537 (1978).

[2] I. Bars and Frederic Green, to be published.

[3] A. Chodos, to be published.

[4] W. A. Bardeen and R. Pearson, Phys. Rev. D 14, 547 (1976).

[5] S. Mandelstam, Bull. Am. Phys. Soc. 22, 541(T) (1977).

[6] K. Wilson, Phys. Rev. D 10, 2445 (1974); J. Kogut and L. Suskind, Phys. Rev. D 11, 395 (1975).