PHYSICAL REVIEW LETTERS

VOLUME 40 | 13 MARCH 1978 | NUMBER 11

Detailed Time-Dependent Description of Tunneling Phenomena Arising from Stochastic Quantization

Kunio Yasue

Department of Physics, Nagoya University, Nagoya 464, Japan

(Received 22 December 1977)

By adopting Nelson's stochastic framework of quantum mechanics (i.e., the stochastic quantization), a time-dependent description of tunneling phenomena is developed. As a simple model, tunnel effect of a one-dimensional quantum mechanical system with bistable potential is analyzed in much detail.

One often encounters the problem of barrier penetration in quantum mechanical systems, e.g., in the NH_3 molecule, in the α decay of nucleus, in the Esaki diode, and in the recent gauge field theory. To describe those barrier penetration phenomena (i.e., tunnel effects), a time-independent treatment with WKB (Wentzel-Kramers-Brillouin) approximation has been adopted. However, in view of the vacuum instability problem in the recent gauge field theory,[1] a detailed time-dependent treatment of the tunnel effect seems to be needed.

In the present Letter, I develop such a time-dependent description of tunnel effects in a fully quantum mechanical fashion. My approach is based on Nelson's stochastic framework of quantum mechanics,[2] which is frequently called *stochastic quantization.*[3]

To simplify the analysis, I restrict myself to the dynamical system with only one degree of freedom. Consider such a dynamical system with Lagrangian

$$L = \dot{x}^2/2 - V(x). \tag{1}$$

This represents a one-dimensional motion of a particle with unit mass under the influence of the potential $V(x)$. Let $V(x)$ be a bistable potential with two relative minima $x=\alpha$ and $x=\beta$ (see Fig. 1). The most familiar example is the anharmonic one:

$$V(x) = ax^2 + bx^3 + cx^4, \quad c>0. \tag{2}$$

In classical mechanics, α and β correspond to two inequivalent vacuum states. If the energy of the system does not exceed the local barrier height $V(\gamma)$, transition between two vacua cannot occur. In quantum mechanics, however, it does occur because of the tunnel effect. What I shall

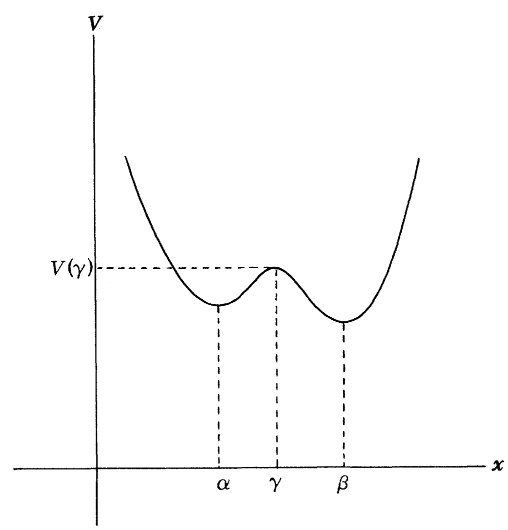

FIG. 1. The bistable potential $V(x)$.

analyze, in the following, is this quantum mechanical tunneling phenomena.

Assume the system to be in an energy eigenstate with energy eigenvalue $E < V(\gamma)$. A wave function of the system $u_E(x) \in L_2(R)$ satisfies the Schrödinger equation

$$-(\hbar^2/2)u_E''(x) + V(x)u_E(x) = Eu_E(x), \tag{3}$$

where $\hbar$ denotes Planck's constant divided by 2π and $'$ means differentiation with respect to x.

Within the conventional framework of quantum mechanics, the wave function $u_E(x)$ does not tell us any information about the details of tunneling except the ratio of probability density $|u_E(\alpha)|^2/|u_E(\beta)|^2$. However, in the framework of stochastic quantization, it tells us that a *quantized motion of the system in the energy eigenstate* u_E *is subjected to a Markov process* $x(t)$ described by a stochastic differential equation

$$dx(t) = a(x(t))\,dt + dw(t). \tag{4}$$

Here, $a(x) = \hbar u_E'(x)/u_E(x)$ and $w(t)$ denotes a Wiener process with diffusion coefficient $\hbar/2$. Therefore, details of the tunneling between two inequivalent classical vacua α and β can be given by obtaining a transition-probability law of the Markov process $x(t)$. Hereafter, I consider the tunneling from α to β.

A transition probability density $p(x,t|y,u)$, with $t > u$, of the Markov process $x(t)$ is known to be an elementary solution of the Fokker-Planck equation[4]

$$\frac{\partial p}{\partial t} = -\frac{\partial[a(x)p]}{\partial x} + \frac{\hbar}{2}\frac{\partial^2 p}{\partial x^2}. \tag{5}$$

Introducing a relative transition probability density $f(x,t|y,u)$ by

$$p(x,t|y,u) = u_E(x) f(x,t|y,u) u_E^{-1}(y). \tag{6}$$

we can transform Eq. (5) into a self-adjoint form[5]

$$-\hbar\frac{\partial}{\partial t}f = \left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + V(x) - E\right]f. \tag{7}$$

It is worthwhile to notice that the Markov process $x(t)$ has a stationary distribution $|u_E(x)|^2$. Since such a Markov process can not traverse each node of the stationary distribution,[2] tunneling through the local potential barrier γ occurs only when the wave function of the system $u_E(x)$ has no nodes within the barrier region. I consider such a case of energy eigenstate.[6]

An elementary solution of Eq. (7) is given by the Wiener integral[7]

$$f(x,t|y,u) = \exp[E(t-u)/\hbar]\int_{\Omega(\substack{x\\t}|\substack{y\\u})}\exp[-\textstyle\int_u^t V(\xi(s))\,ds/\hbar]\mu_W(d\xi), \tag{8}$$

where μ_W denotes a Wiener measure with diffusion coefficient $\hbar/2$ and $\Omega(\substack{x\\t}|\substack{y\\u})$ a totality of continuous paths ξ's with $\xi(u) = y$ and $\xi(t) = x$. Thus the transition probability density of the Markov process $x(t)$ can be written as

$$p(x,t|y,u) = [u_E(x)/u_E(y)]\exp[E\cdot(t-u)/\hbar]\int_{\Omega(\substack{x\\t}|\substack{y\\u})}\exp[-\textstyle\int_u^t V(\xi(s))\,ds/\hbar]\mu_W(d\xi). \tag{9}$$

Equation (9) gives us a detailed time-dependent description of the tunneling between the classical vacua α and β. Namely, $p(\beta,t|\alpha,u)$ *represents a tunneling probability of the system.* Equation (9) reduces to the well-known WKB prescription if we pass to the semiclassical limit[8]:

$$p(\beta,t|\alpha,u) \propto [u_E(\beta)/u_E(\alpha)]\exp[-\hbar^{-1}\textstyle\int_u^t[\frac{1}{2}\dot{\xi}(s)^2 + V(\xi(s)) - E]\,ds]_{\max}$$

$$\propto \exp\left[-(2/\hbar)\int_\alpha^\beta\{2[V(\xi)-E]\}^{1/2}d\xi\right]_{\max}, \tag{10}$$

where $[\;]_{\max}$ means to take a maximum value.

Notice that I have succeeded in obtaining the "Euclidean" prescription of tunneling phenomena mainly used in the gauge field theory[1] and other fields.[9] In the semiclassical limit, tunneling occurs along a classical Euclidean path which minimizes the Euclidean action integral in the exponent.

Now what is left for us is to compute the Wiener integral (9) and to obtain the tunneling probability. Let $\{u_n(x)\}_{n=0}^{\infty} \subset L_2(R)$ be a complete normalized orthogonal system of eigenfunctions of the Hamiltonian;

$$-(\hbar^2/2)u_n''(x) + V(x)u_n(x) = E_n u_n(x), \quad n = 0, 1, \ldots. \tag{11}$$

Then, the tunneling probability between the classical vacua α and β becomes

$$p(\beta,t|\alpha,u)=[u_E(\beta)/u_E(\alpha)]\sum_{n=0}^{\infty}u_n(\beta)u_n(\alpha)\exp[-(E_n-E)(t-u)/\hbar]. \quad (12)$$

For sufficiently large $t-u$, we can approximate Eq. (12) by taking only one counter term into account, obtaining

$$p(\beta,t|\alpha,u)\simeq[u_E(\beta)u_0(\beta)u_0(\alpha)/u_E(\alpha)]\exp[(E-E_0)(t-u)/\hbar]. \quad (13)$$

Thus we have found that the penetration rate of the system with energy $E<V(\gamma)$ from the "false" vacuum α to the "true" one β is given by $(E-E_0)/\hbar$, which can be computed immediately in each practical case.

I conclude with the following comments. (i) Quantized motion of the system in the energy eigenstate is subjected to a Markov process. (ii) Tunneling probability of the system with energy $E<V(\gamma)$ through the local potential barrier γ is given by the transition probability density of the Markov process. (iii) In the semiclassical limit, the tunneling probability thus obtained reduces to the well-known WKB prescription. (iv) Penetration rate through the local potential barrier γ is proportional to the energy of the system measured from the lowest energy eigenvalue. (v) My approach to quantum mechanical tunneling phenomena has a close analogy with Langer's analysis[10] of the nucleation process in classical statistical mechanics. (vi) My formulation is valid not only for one-dimensional systems but also for higher-dimensional ones.

[1]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); P. H. Frampton, Phys. Rev. Lett. 37, 1378 (1976); S. Coleman, Phys. Rev. D 15, 2929 (1977).

[2]E. Nelson, Phys. Rev. 150, 1079 (1966), and *Dynamical Theories of Brownian Motion* (Priceton Univ. Press, Princeton, N. J., 1967).

[3]F. Guerra and P. Ruggiero, Phys. Rev. Lett. 31, 1022 (1973); F. Guerra, in *C*-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory*, edited by D. Kastler (North-Holland, Amsterdam, 1976).

[4]J. L. Doob, *Stochastic Processes* (Wiley, New York, 1953).

[5]V. E. Benes and L. A. Shepp, Theor. Prob. Appl. 13, 475 (1968); R. I. Cukier, K. Lakatos-Lindenberg, and K. E. Shuler, J. Stat. Phys. 9, 137 (1973); H. Tomita, Prog. Theor. Phys. 55, 960 (1976).

[6]The oscillation theorem asserts that the lowest energy eigenstate of a one-dimensional system has no nodes anywhere and the nth excited state has n nodes. Concerning the location of nodes, I need a further consideration of symmetry. See, for example, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1965).

[7]E. Nelson, J. Math. Phys. (N.Y.) 5, 332 (1964).

[8]K. Yasue, to be published.

[9]K. Maki, Phys. Rev. Lett. 39, 46 (1977).

[10]J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967).

Quantum Nondemolition Measurements of Harmonic Oscillators

Kip S. Thorne, Ronald W. P. Drever,[a] Carlton M. Caves,
Mark Zimmermann, and Vernon D. Sandberg
California Institute of Technology, Pasadena, California 91125
(Received 23 January 1978)

The complex amplitude $X_1+iX_2\equiv(x+ip/m\omega)e^{i\omega t}$ of a harmonic oscillator is constant in the absence of driving forces. Although the uncertainty principle forbids precise measurements of X_1 and X_2 simultaneously ($\Delta X_1\Delta X_2\geq\hbar/2m\omega$), X_1 alone can be measured precisely and continuously ("quantum nondemolition measurement"). Examples are given of measuring systems that do this job. Such systems might play a crucial role in gravitational-wave detection and elsewhere.

A standard technique for measuring very weak forces is to let them act on a high-Q harmonic oscillator, and then to monitor the motion of the oscillator.[1] Examples are Dicke-Eötvös experiments and gravitational-wave detectors.[1] Some future gravitational-wave detectors may use massive ($m\sim100$ kg) dielectric crystals (sapphire or silicon) with eigenfrequencies $\omega/2\pi\sim5000$ Hz, cooled to a few millidegrees where their fundamental modes would contain $N=kT/\hbar\omega\sim10^4$ quanta

