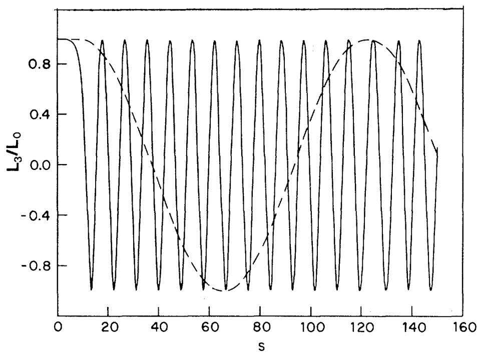

FIG. 2. $L_3(x,t)$ as a function of dimensionless time $s = \frac{1}{2} g l_0^{1/2} t$. Full line, $x = \frac{1}{4}\lambda/20$; dashed line, $x = \frac{1}{4}\lambda$.

the former rate, and occurs at a later time $s = 12.35$, with $n/l_0 = 0.24$. From Fig. 2 one sees that $L_3(x = \frac{1}{4}\lambda, s)$ decays slowly initially and then has faster, almost sinusoidal oscillations; these oscillations are stimulated by the field due to emission by AMO's at other points in the cavity. Chang and Stehle[7] have studied the interaction of TLS's with the electromagnetic field between partially reflecting mirrors, applying a generalized Weisskopf-Wigner[14] method. They predict an exponential decay of the TLS's even in the limit of totally reflecting mirrors, which is contrary to present results and also to the reasonable expectation that the de-excited TLS's will reabsorb the photons when the radiation density in the cavity is high enough. Results for longer times and an analytic approximation for the case of "continuous" distribution of AMO's in the cavity ($M \to \infty$) are given elsewhere.[13]

It is a pleasure to thank Professor I. R. Senitzky for many illuminating discussions.

(a) Present address: Department of Theoretical Chemistry, University of Bristol, Bristol, England.

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Significance of *K* Distributions in Scattering Experiments

E. Jakeman and P. N. Pusey
Royal Signals and Radar Establishment, Malvern, WR14 3PS, United Kingdom
(Received 14 December 1977)

Evidence is presented which suggests that a class of modified Bessel-function distributions may have special significance in describing the statistics of radiation scattered by media characterized by a wide range of length scales. It is shown that these distributions may be obtained mathematically by applying a limit procedure to the random-walk problem with a variable number of steps. The choice of distribution for the step-number fluctuations is briefly discussed.

In an experiment involving the scattering of electromagnetic radiation the scattered field $E(\vec{r},t)$ at detection point $\vec{r}$ and time t can frequently be represented as the vector sum of independent contributions from a number N of scattering centers within the scattering medium. Systems for which this approach has proven useful include not only collections of discrete scatterers such as particles in suspension[1,2] or seed particles in fluid flows[3] but also continuous systems of the deep phase screen type such as very rough surfaces or thin layers of highly turbulent media which contain focusing elements.[4,5] When N is large the complex field $E(\vec{r},t)$ is Gaussian distributed by virtue of the central limit theorem; the envelope $|E|$ is then Rayleigh distributed and the intensity $I=|E|^2$ has a negative exponential distribution. In many cases of practical interest, however, N is not large and departures from Gaussian statistics are found.[6] One may deliberately choose to work in the "non-Gaussian regime," as in some laser-light-scattering experiments, or it may be unavoidable, as in the case of a high-resolution microwave radar operating over a rough sea.[7] Previously we suggested that a class of distributions based on the modified Bessel functions K_ν might prove useful in this latter context but data taken recently in a wide variety of experiments suggest that these distributions may be more generally applicable in non-Gaussian situations. In this Letter, after introducing K distributions we present further experimental evidence in support of this conjecture. We then show that K distributions can be obtained by applying a special limit prcedure in the random-walk problem.

We write the scattered field

$$E(\vec{r},t)=e^{i\omega t}\sum_{i=1}^{N}a_i(\vec{r},t)\exp[i\varphi_i(\vec{r},t)]$$
$$\equiv e^{i\omega t}A(\vec{r},t)e^{i\Phi(\vec{r},t)}, \tag{1}$$

where ω is the "carrier frequency" of the incident radiation, $a_i(\vec{r},t)$ is a real "form factor" governing the angular distribution of radiation from the ith scatterer, and $\varphi_i(\vec{r},t)$ is a phase factor depending on its position at time t with respect to the observation point $\vec{r}$. We assume (i) that a_i and φ_i are random variables which are statistically independent from each other and from a_j and φ_j for all $i\neq j$, and (ii) that the scatterers introduce path differences exceeding the wavelength of the radiation so that the φ_i can be regarded as being uniformly distributed over 2π (strong scattering). Equation (1) then describes a random walk of a finite number of steps of fluctuating lengths in the complex E plane. The distribution of the envelope $A(\vec{r},t)$ can be determined by Fourier transformation of the characteristic function

$$C(\lambda)=\int e^{i\vec{\lambda}\cdot\vec{E}}P(\vec{E})\,d^3E\equiv\langle e^{i\vec{\lambda}\cdot\vec{E}}\rangle. \tag{2}$$

After angular integration we get

$$C(\lambda)=\langle J_0(\lambda A)\rangle=\langle J_0(\lambda a)\rangle^N, \tag{3}$$

where $\langle\rangle$ here indicate averages over the distribution $p(a)$ of fluctuating amplitude factors. (Distributions satisfying this equation are, by definition, infinitely divisible.[8])

The limiting distribution for $N\to\infty$ may be obtained in a simple way by scaling the amplitudes $\{a_i\}$ with $\sqrt{N}$ so that the mean intensity $\langle A^2\rangle=N\langle a^2\rangle$ remains finite:

$$\lim_{N\to\infty}\langle J_0(\lambda a/\sqrt{N})\rangle^N=\exp(-\lambda^2\langle a^2\rangle/4), \tag{4}$$

leading to a Rayleigh distribution for A, independent of the statistics of the $\{a_i\}$. This is just the central-limit theorem result expressing the fact that the real and imaginary parts of the complex field [Eq. (1)] are independent Gaussian variables for large N. Note that when the $\{a_i\}$ are themselves Rayleigh distributed then so too is the envelope A *even for finite* N. This is because the Rayleigh distribution is *stable* or a *fixed point* with respect to the convolution, Eq. (3).[8]

In general, however, the distribution defined by Eq. (3) is not Rayleigh for finite N and may well describe a signal which fluctuates more wildly than the noiselike behavior obtained for large N. This situation of enhanced fluctuations is expected to obtain when the region of scattering medium contributing to the received field is comparable to or smaller than the largest structure present, with N denoting the number of scattering centers within this region. Complete solution of the problem would then involve (i) calculating the probability distribution $p(a)$ for the situation of interest, and (ii) evaluating Eq. (3) and inverting Eq. (2). While (i) is in principle possible, if difficult, it appears that (ii) is not possible analytically for arbitrary $p(a)$. In searching for a model for non-Gaussian statistics we discovered a class of distributions with attractive properties based on the modified Bessel functions K_ν for which Eq. (3) can be inverted analytically.[7] We take

$$p(a)=\frac{2b}{\Gamma(1+\nu)}\left(\frac{ba}{2}\right)^{\nu+1}K_\nu(ba), \quad \nu>-1. \tag{5}$$

Then, from Eq. (3), envelope A is also K-distributed with increased index

$$\nu_N=(\nu+1)N-1. \tag{6}$$

Although these two-parameter distributions are infinitely divisible with respect to the convolution, Eq. (3), they are not stable, approaching a Rayleigh distribution as N becomes large, i.e., as $\nu_N \to \infty$. Their moments always lie between those of a Rayleigh distribution and those of a log-normal distribution with the same mean and variance.[7]

Although the K distributions are clearly interesting from a mathematical point of view and possibly useful as a model for non-Gaussian statistics, no real physical basis has so far been given for the choice of $p(a)$ made in the preceding discussion. It was somewhat to our surprise, therefore, to find experimentally that the radiation scattered by a variety of systems is K-distributed to a significant degree of accuracy. These experiments include the scattering of laser light by (i) a turbulent layer of nematic liquid crystal,[9,10] (ii) a turbulent layer of air,[11] (iii) a turbulent layer of water,[12] and (iv) an extended region of atmospheric turbulence,[13] (v) the scattering of starlight by the upper atmosphere[14] (stellar scintillation), and (vi) the scattering of microwave radiation by a small area of rough sea.[15] Only in cases (ii) and (vi) have the experimental data been compared with K distributions in the open literature. In the limited space available here we present a reanalysis of old data for case (i) which were previously analyzed in terms of a facet model for the wave front emerging from the scattering medium. A thin layer of liquid crystal was driven into a turbulent state by applying an electric field (dynamic scattering). The typical size of a turbulent eddy was a few microns, comparable to the (adjustable) size of a focused laser beam so that the non-Gaussian (or non-Rayleigh) situation was easily achieved. In Fig. 1 we compare the moments $\langle I^n\rangle/\langle I\rangle^n$ of the experimental intensity probability distributions with those of K distributions having the same mean and variance. There is almost perfect agreement to within experimental error for a range of second moments from ~2 (the Gaussian value) to ~10. It should be emphasized that an attractive feature of distributions based on Eq. (1) is their built-in scaling with number of scatterers N [through Eq. (6) for K distributions] and therefore with the illuminated volume of scattering medium. This scaling was accurately verified in the liquid crystal experiments by varying the size of the laser spot.[10]

The applicability of K distributions to such a wide range of experimental data led us to search for a limiting process leading to them rather than to the Rayleigh distribtuion as in Eq. (4). We find that this can be achieved by introducing number fluctuations into the random walk problem. Suppose that in a two-dimensional random walk of the type of Eq. (1) the number N of contributions to the scattered field is fluctuating in a way which is statistically independent from the $\{a_i\}$. The convolution, Eq. (3), may then be averaged over the distribution of N. For example, if N is Poisson-distributed we obtain[16,1]

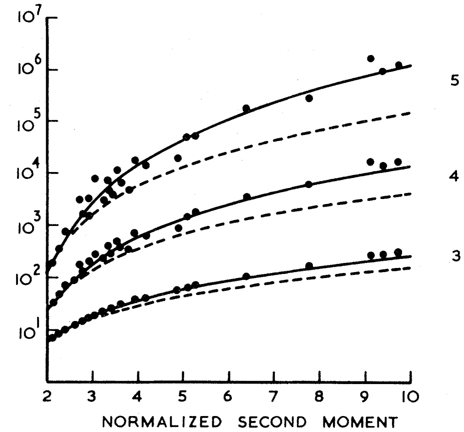

FIG. 1. Higher normalized moments, $\langle I^n\rangle/\langle I\rangle^n$, of the intensity distribution of light dynamically scattered by a liquid crystal as a function of the normalized second moment, $\langle I^2\rangle/\langle I\rangle^2$, for $n = 3$, 4, and 5. The solid lines are the values for K distributions; the dashed lines are the predictions of the old "facet" model.

$$\langle J_0(\lambda A)\rangle = \exp\{\overline{N}[\langle J_0(\lambda a)\rangle - 1]\}.$$

If we now define a limiting process for large $\overline{N}$ in which a is again scaled by $\overline{N}^{1/2}$ we obtain

$$\lim_{\overline{N}\to\infty} \exp\{\overline{N}[\langle J_0(\lambda a \overline{N}^{-1/2})\rangle - 1]\} = \exp(-\lambda^2\langle a^2\rangle/4)$$

leading to a Rayleigh distribution for A as before. This is consistent with the fact that the normalized variance of a Poisson distribution $\propto \overline{N}^{-1}$ so that the problem reduces to the fixed N case as $\overline{N}\to\infty$. However, consider the negative binomial distribution

$$P(N) = \binom{N+\nu}{N} \frac{[\overline{N}/(\nu+1)]^N}{[1+\overline{N}/(\nu+1)]^{N+\nu+1}} \tag{7}$$

The normalized variance of this distribution is given by

$$(\langle N^2\rangle - \langle N\rangle^2)/\langle N\rangle^2 = 1/\overline{N} + \nu + 1 \tag{8}$$

which remains finite as $\overline{N}\to\infty$. Summing the convolution, Eq. (3), over this distribution and scaling a with $\bar{N}^{1/2}$ gives

$$\langle J_0(\lambda A)\rangle = \left[1+\frac{\bar{N}}{1+\nu}-\frac{\bar{N}}{1+\nu}\langle J_0(\lambda a\bar{N}^{-1/2})\rangle\right]^{-(\nu+1)} \tag{9}$$

and clearly,

$$\lim_{\bar{N}\to\infty}\langle J_0(\lambda A)\rangle = \left[1+\frac{\lambda^2\langle a^2\rangle}{4(1+\nu)}\right]^{-(\nu+1)}. \tag{10}$$

A straightforward Bessel transform of Eq. (10) shows that A is K-distributed and we have achieved our objective. Note that the result, Eq. (10), is independent of the statistics of the $\{a_i\}$ as in the central limit theorem.

Thus we have established a new limit theorem which predicts that the amplitude of the resultant of a two-dimensional random walk, with step number varying according to Eq. (7), will be K-distributed as the mean number of steps becomes large. It now remains to justify the assumption of negative binomial number fluctuations [Eq. (7)] as a sensible model for the scattering systems studied. We discuss this using plausibility rather than rigorous arguments. We note that only for a Poisson distribution are "events" uncorrelated; in this case an event is the presence of a scatterer. For a negative binomial distribution the events are correlated and will occur in "bunches." (The distribution is often used to model variable-mean Poisson processes.) This description is not unreasonable for a rough sea surface where the scatterers may be viewed as collections of specular points associated with small wavelets carried on the top of a larger scale structure. Similarly there is a large range of scale sizes in a turbulent medium where a large eddy may break down into a number of smaller ones. The negative binomial distribution is, in fact, an exact stationary solution for the population in a simple birth-death-immigration process[17] which could well describe the evolution of eddies in a defined region of a turbulent medium.

Obviously fluctuations in the number of scattering centers will only be correlated over distances less than the maximum length scale in the medium (the outer scale of turbulence). Restricting ourselves to a thin scattering layer for simplicity, we see that the arguments of the previous paragraph only apply *directly* when the area A_s of scattering medium seen by the detector is smaller than or comparable to the maximum correlation area A_c. However, if $A_s \gg A_c$, then the field at the detector will be the sum of the essentially independent fields scattered by each elemental area A_c. Thus the medium may be regarded as being made up of a number of *independent* "superscatterers" of area $\sim A_c$ each of which consists of *correlated* groups of lesser scatterers. By the arguments developed above, the amplitude scattered by a single superscatterer will be K-distributed. Therefore this approach, if correct, justifies the use of Eq. (1) with $p(a)$ given by a K distribution and with $N\sim A_s/A_c$ as the number of superscatterers. In this picture the amplitude A will be K-distributed for values of A_s/A_c ranging from $\ll 1$ to $\gg 1$ (provided $A_s^{1/2}$ is much less than the smallest correlation length in the medium —the inner scale of turbulence.)

It is worth noting that, although our derivation of the K distributions is based on a two-dimensional random walk, Eq. (1), these are also obtained by the same argument in the one-dimensional case

$$X=\sum_{i=1}^{N}x_i$$

if the steps x_i can be either positive or negative. On the other hand, if the steps are restricted to be positive then the number distribution (7) leads to the result

$$\lim_{\bar{N}\to\infty}\langle e^{i\lambda X}\rangle = [1-i\lambda\langle x\rangle]^{-(\nu+1)} \tag{11}$$

corresponding to a gamma distribution (which also possesses the property of infinite divisibility[18]). If we identify X and x_i with intensities then (11) gives the limit distribution for an incoherent scattering experiment in which intensities are additive, rather than fields as in Eq. (1). Experiments of this type together with measurements of other properties of the scattered field such as the temporal autocorrelation function of the intensity could possibly be used to test the ideas we have presented and may also provide a means for determining the outer scale of turbulence.

In conclusion, we would like to emphasize that limit distributions in the random walk problem constitute an entire field of study in themselves[18] and that, inevitably, the simple analysis presented here has been obtained at the expense of mathematical rigor. We have demonstrated, however, that number fluctuations may be important in this context, and have established a need to investigate their role further. This may shed new light on the problems of turbulence and critical phenomena in view of the increasing use in this area of research of renormalization group techniques which are intimately related to limit theorems in probability theory[19].

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Diffusion and Scattering of Test Particles in a Turbulent Plasma

R. L. Stenzel and W. Gekelman

Department of Physics, University of California, Los Angeles, California 90024

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A small-diameter, low-energy, test electron beam is injected along $\vec{B}_0$ into a collisionless plasma in which current-driven ion sound turbulence can be generated. The scattering of the test particles across $\vec{B}_0$ due to fluctuating fields $\vec{E}_w \perp \vec{B}_0$ gives a direct measure for the turbulent spatial diffusion coefficient $D_{e\perp}$. Investigation of the velocity-space diffusion shows strong pitch-angle scattering effects. No significant anomalous resistivity is observed.

Although ion sound turbulence has been studied by many authors[1] the direct interplay of particles and waves has rarely been observed directly. In this Letter we describe an experiment where test particles (electrons) are injected into a turbulent plasma and subsequently followed in real and velocity space. Diffusion coefficients and resistivity are obtained in a case where the turbulence spectrum $\omega(\vec{k})$ has been carefully analyzed. We show that Bohm-like diffusion arises from random $\vec{B}_w \times \vec{B}_0$ drifts due to perpendicular-wave electric fields, $\vec{E}_w$.

The experiment is performed in a large, magnetized, nearly collisionless discharge plasma[2] of parameters $n_e \simeq 10^9$ cm^{-3}, $T_e \simeq 10 T_i \simeq 2$ eV, $B_0 \simeq 130$ G, $\nu_i/\omega_{pi} \simeq 10^{-3}$, Ar and He. As is schematically shown in Fig. 1(a), the uniformly magnetized plasma column is divided by a fine wire mesh into two regions: the experimental section in which a field-aligned current is drawn to an end anode, and the source region of higher density which supplies the electrons to maintain the current. Aside from the use of Langmuir probes and resonance cones, the diagnostics include movable rf probes to perform three-dimensional cross-correlation measurements, and a test electron-beam source which projects a low-energy (1–10 eV), low-density ($n_b/n_e \ll 1$), pencil beam (2 mm^2) along $\vec{B}_0$. In order to distinguish the test electrons from the background electrons the beam is weakly velocity modulated ($f_m \simeq 50$ kHz $\ll f_{pi}$) and resonantly detected, and its relative distribution is displayed versus probe voltage referenced to ground.

By drawing a field-aligned electron current to the end anode, ion acoustic waves are driven un-