tively, by forcing coefficients a_0B_0, $-a_1B_1$, a_2B_2, and $-a_3B_3$ when inserted into (13) to yield a curve B' in Fig. 2 within experimental uncertainty of measured curve B. The polarization distribution obtained from the determined coefficients, assuming $a_1 = a_2 = a_3 = 2a_0$, is the distribution b in the inset of Fig. 1.

The samples discussed in this Letter were chosen to illustrate the theory. Other copolymer samples and homopolymer (polyvinylidene fluoride) samples under different poling conditions can exhibit nearly uniform polarization.[9] In such cases, the response is nearly steplike and $B_n/B_0 = 0$ for all n.

The present work indicates the source of ambiguity in the Collins's deconvolution procedure. The thermal pulse data (under conditions similar to those in Collins's experiments) yield no more than ten or fifteen coefficients [based on $N = (\tau_1/t_r)^{1/2}$]. Collins's electrical analog sought to obtain discrete distributions characterized by twenty adjustable parameters. Any discrete distribution (of which there are many) consistent with the determinable Fourier coefficients would reproduce the measured transients within the noise in the data.

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Scaling Theory of the Asymmetric Anderson Model

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A scaling theory is used to show that for temperatures $T \ll U$, the properties of the asymmetric Anderson model ($U \gg |E_d|, \Delta$) are universal functions of the scaling invariants Δ and $E_d^* = E_d + (\Delta/\pi)\ln(W_0/\Delta)$, where W_0 is the conduction electron bandwidth or U, whichever is smaller. Crossovers between various regimes of simple behavior as the temperature changes are described. $|E_d^*| \lesssim \Delta$ is identified as the criterion for a "mixed-valence" ground state, where the susceptibility $\approx \Delta^{-1}$. For $-E_d^* \gg \Delta$, there is a local-moment regime with a Kondo temperature $T_K \approx \Delta \exp(\pi E_d^*/2\Delta)$.

There has been recent interest in the asymmetric Anderson model[1] ($U \gg |E_d|, \Delta$) in connection with the theory of "mixed-valence" rare-earth materials.[2-4] The numerical renormalization-group technique pioneered by Wilson[5] allows the thermodynamic properties to be calculated,[3] but the parameter space is large. Analytic results can clarify the dependence of physical properties on the model parameters, and provide a framework for the numerical exploration of the "crossovers" between limits describable by a simple effective Hamiltonian. This Letter reports a *scaling property* of the asymmetric Anderson model; that is, universality of model properties as functions of the scaling invariants Δ and $E_d^* = E_d + (\Delta/\pi)\ln(W_0/\Delta)$, where $W_0 \approx U$ or the conduction electron bandwidth, whichever is smaller. The scaling equations also allow a simple description of the temperature dependence of physical properties.

The (nondegenerate) Anderson model is characterized by the parameters E_d, U, and $\Delta(\omega)$, and is

$$H = H^0 + E_d\sum_\sigma n_{d\sigma} + Un_{d\uparrow}n_{d\downarrow} + \sum_{k\sigma} V_{kd}\, c_{k\sigma}^\dagger c_{d\sigma} + \text{H.c.},$$
$$H^0 = \sum_{k\sigma}\epsilon_k n_{k\sigma}; \quad \Delta(\omega) = \sum_k |V_{kd}|^2\delta(\omega - \epsilon_k). \qquad (1)$$

$\Delta(\omega)$ is essentially characterized by Δ $[=\Delta(0)]$, and a bandwidth W where $\Delta(\omega)\approx\Delta$ for $|\omega|\ll W$ and $\Delta(\omega)\approx 0$ for $|\omega|\gg W$. The limit $U\gg\Delta$ may be investigated by a perturbation expansion in Δ. If $W\gg U$ the expansion is independent of W, but for $U\gg W\gg|E_d|$, logarithmic dependence on W appears in each order; for $W\gg T$, the leading terms in the expansion for the impurity susceptibility are

$$\chi=\frac{1}{6T}\left[1+\frac{\Delta}{3\pi T}\ln\left(\frac{T}{W}\right)+\dots\right]\quad(T\gg|E_d|),\quad(2)$$

$$\chi=\frac{\Delta}{2\pi E_d^{\,2}}\left[1+\frac{2\Delta}{\pi E_d}\ln\left(\frac{E_d}{W}\right)+\dots\right]\quad(E_d\gg T),\quad(3)$$

$$\chi=\frac{1}{4T}\left[1+\frac{2\Delta}{\pi E_d}+\frac{1}{2}\left(\frac{2\Delta}{\pi E_d}\right)^2\ln\left|\frac{T^2}{WE_d}\right|+\dots\right]\quad(-E_d\gg T).\quad(4)$$

These expansions are ultraviolet divergent as $W\to\infty$; however, for $W\gg U$, they are given by equivalent expressions where a quantity of order U replaces W. (The effects of processes involving high-energy conduction-band states cancel exactly in the noninteracting limit $U=0$; this cancellation remains for conduction band energies $\gg U$.) Perturbation theory indicates that states with energies in the range $|E_d|, T\ll|\omega|\ll U$ play an important role in low-energy processes as virtually excited intermediate states.

Logarithmic dependence on a high-energy cutoff is the hallmark of a *scaling property*, where the physics at low energies depends not on the "bare" parameters, but on renormalized parameters that take into account the effect of high-energy intermediate states. Such quantities may be identified as the invariants of a *scaling transformation:* If the cutoff W is reduced to $W-|dW|$ by integrating out states with energies $W-|dW|<|\omega|<W$, the bare parameters are renormalized, but the low-energy physics is unchanged, and thus depends on the *scaling invariants*. Of course, such a truncation of the conduction band not only renormalizes the bare parameters, but also generates both new couplings and retardation.[6] However, retardation should not affect processes with energies $|\omega|\ll W$, and the new couplings should be "irrelevant" in that they vanish in the limit $W\to\infty$. In this limit, a truncation procedure that generates a new effective Hamiltonian with renormalized parameters does so as a consequence of an intrinsic scaling property of the model.

The scaling equations may be derived in a manner reminiscent of Anderson's "poor man's" treatment[7] of the Kondo problem. Divide $\Delta(\omega)$ into $\Delta((1+\lambda)\omega)-\lambda\omega\Delta'(\omega)$, where λ is a positive infinitesimal, and $\Delta'(\omega)$ is the derivative. The positive quantity $-\lambda\omega\Delta'(\omega)$ represents the contribution to $\Delta(\omega)$ from high-energy states which are to be integrated out, preserving the *form* of $\Delta(\omega)$, but changing its *scale*. If $U\gg W\gg|E_d|$, hybridization with these states renormalizes the d-orbital states $|0\rangle$ and $|1\sigma\rangle$, but the state $|2\rangle$ is decoupled from the conduction band. Particle states in the cutoff region are labeled k^+, hole states k^-; to lowest order in λ $(=-d\ln W)$ the transformation is

$$|\tilde{0}\rangle=|0\rangle-\sum_{k=k^-,\sigma}\frac{V_{kd}{}^*}{|\epsilon_k|+E_1-E_0}c_{k\sigma}|0\rangle,\quad(5)$$

$$\tilde{E}_0=E_0-\sum_{k=k^-,\sigma}\frac{|V_{kd}|^2}{|\epsilon_k|+E_1-E_0},\quad(6)$$

$$|1\tilde{\sigma}\rangle=|1\sigma\rangle-\sum_{k=k^+}\frac{V_{kd}}{|\epsilon_k|+E_0-E_1}c_{k\sigma}{}^\dagger|0\rangle,\quad(7)$$

$$\tilde{E}_1=E_0-\sum_{k=k^+}\frac{|V_{kd}|^2}{|\epsilon_k|+E_0-E_1}.\quad(8)$$

The scaling equation for E_d $(\equiv E_1-E_0)$ is thus

$$\frac{dE_d}{d\ln W}=\frac{1}{\pi}\int_0^\infty d\omega\left(\frac{2\omega\Delta'(-\omega)}{\omega+E_d}+\frac{\omega\Delta'(\omega)}{\omega-E_d}\right).\quad(9)$$

Since $\Delta'(\omega)\approx 0$ unless $|\omega|\approx W$, (9) simplifies to

$$dE_d/d\ln W=-\Delta(0)/\pi+O(E_d\Delta(W)/W).\quad(10)$$

The transformation of $\Delta(0)$ is found from the renormalization of V_{kd} when $\epsilon_k=0$:

$$\tilde{V}_{kd}=\langle\tilde{0}|c_{k\sigma}H|1\tilde{\sigma}\rangle(\langle\tilde{0}|\tilde{0}\rangle\langle 1\tilde{\sigma}|1\tilde{\sigma}\rangle)^{-1/2},\quad(11)$$

$$d\Delta(0)/d\ln W=O(\Delta(0)\Delta(W)/W).\quad(12)$$

In the limit $W\to\infty$, the right-hand side of (12) vanishes, and $\Delta(0)$ is unrenormalized; however, (10) is nontrivial, and E_d is strongly renormalized by scaling. This is because the state $|0\rangle$ can hybridize with both $c_{k\uparrow}|1\uparrow\rangle$ and $c_{k\downarrow}|1\downarrow\rangle$, while (as $|2\rangle$ is decoupled) $|1\sigma\rangle$ only mixes with $c_{k\sigma}{}^\dagger|0\rangle$; E_0 is thus reduced by twice as much as E_1, and E_d rises as scaling proceeds. This feature is absent if $W\gg U$; a similar derivation in that case produces no nontrivial scaling equations. It would be pointless to be more precise about the terms $O(\Delta(W)/W)$; not only are they "irrelevant" (when they are significant, so are new couplings and retardation), but they depend on the detailed form of $\Delta(\omega)$ when $|\omega|\simeq W$—note that the nontrivial term in (10) involves $\Delta(0)$, independent of its form in the cutoff region. Essentially equivalent equations have recently been independently reported by Jefferson,[4] though they are phrased in somewhat different notation, and include "irrelevant" terms derived using a particular cutoff prescription.

The scaling invariant obtained by integrating (10) is $E_d^* = E_d + (\Delta/\pi)\ln(W_0/\Delta)$, where E_d and W_0 are the initial or "bare" values.[8] (If initially $W \gg U$, renormalization of E_d only begins when W has been scaled down to $W \approx W_0 \approx U$.) As W is reduced, E_d rises along the *scaling trajectory*

$$E_d(W) = E_d^* - (\Delta/\pi)\ln(W/\Delta). \quad (13)$$

The scaling trajectories (13) are plotted in Fig. 1.

The scaling laws were derived assuming that particle states in the cutoff were empty and hole states full. For $W < T$, the scaling laws change,[9] and further scaling produces no renormalization of E_d. For $T \gg \Delta$, the physics is essentially atomic, and is thus described by a free orbital with a temperature-dependent level $E_d(T)$, given by setting $W = T$ in (13).

For $E_d^* \gg \Delta$, scaling stops when $W \simeq E_d(W) \simeq T^*$ $(\gg \Delta)$, where

$$T^* + (\Delta/\pi)\ln(\alpha T^*/\Delta) = E_d^*; \quad (14)$$

α $[\approx O(1)]$ is a universal number characteristic of the crossover.[10] Reducing W below T^* produces no further renormalization[9] since the states $|1\sigma\rangle$ become decoupled from low-energy processes. At temperatures below T^*, charge fluctuations are frozen out, $\langle n_d\rangle \approx 0$, and the impurity susceptibility is given by perturbation theory:

$$\chi(T) = \frac{1}{4T}\left(\frac{2\exp(-T^*/T)}{1+2\exp(-T^*/T)}\right) + \frac{\Delta}{2\pi T^{*2}}. \quad (15)$$

Below some temperature T_{FL}, (15) is dominated by the second, temperature-independent term, characteristic of a Fermi liquid. The effective Curie constant $T\chi$ is shown schematically as a function of temperature in Fig. 2(a). It is a measure of the effective degeneracy of the impurity orbital, and rises from $\frac{1}{8}$ (fourfold degeneracy) for $T \gg U$, to $\frac{1}{6}$ (triplet) for $U \gg T \gg T^*$; below T^* it falls to zero (singlet), becoming linear in the Fermi-liquid regime below T_{FL}.

When $|E_d^*| \lesssim \Delta$, the orbital retains effective triplet degeneracy till the crossover temperature $T \approx \Delta$ when irrelevent terms grow and scaling breaks down. The system goes directly into a Fermi-liquid regime; $\langle n_d\rangle$ remains substantially nonintegral at $T = 0$, and this regime may be de-

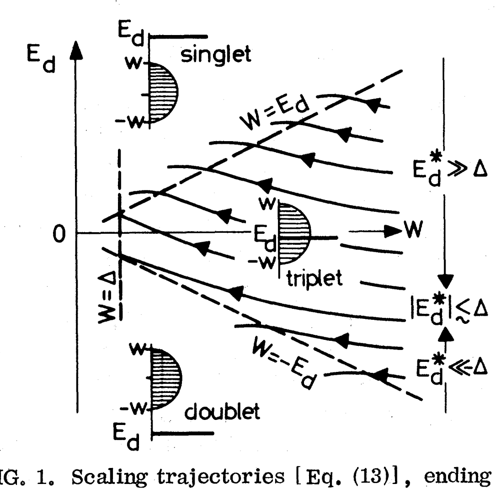

FIG. 1. Scaling trajectories [Eq. (13)], ending at crossovers (broken lines) to a singlet regime ($E_d > W$) for $E_d^* \gg \Delta$, to a doublet local-moment regime ($E_d < -W$) for $E_d^* \ll -\Delta$, and to a mixed-valence Fermi-liquid regime for $|E_d^*| \lesssim \Delta$.

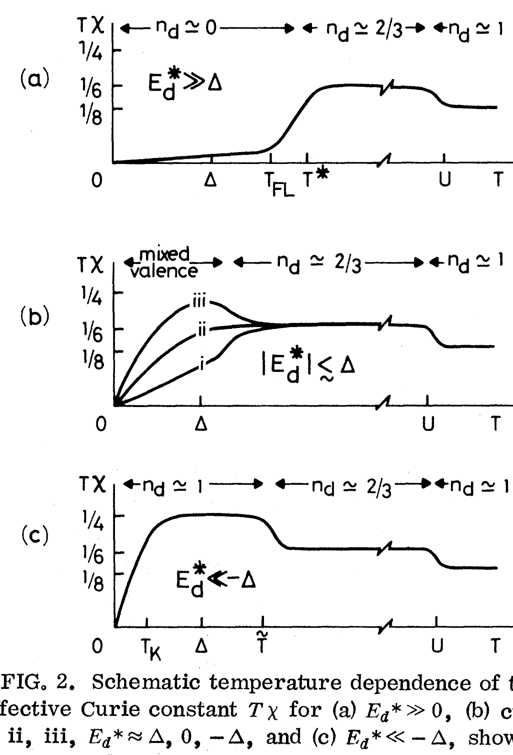

FIG. 2. Schematic temperature dependence of the effective Curie constant $T\chi$ for (a) $E_d^* \gg 0$, (b) curves i, ii, iii, $E_d^* \approx \Delta$, 0, $-\Delta$, and (c) $E_d^* \ll -\Delta$, showing crossovers between atomic regimes where the effective degeneracy is fourfold ($T\chi = \frac{1}{8}$), triplet ($T\chi = \frac{1}{6}$), doublet ($T\chi = \frac{1}{4}$), and singlet ($T\chi = 0$). At low temperatures there is a crossover to a Fermi-liquid regime where $T\chi$ is linear in T. Note that though the temperature scale is drawn linearly to emphasize this Fermi-liquid behavior, T_K, Δ, $\tilde{T}$, and T^* indicate *scales* of temperature which may differ by many orders of magnitude.

scribed as one of "mixed valence." (Note that $|E_d^*| \lesssim \Delta$ is the criterion for mixed valence, *not* $|E_d| \lesssim \Delta$ as commonly supposed[2-4]; depending on W_0, E_d^* may be arbitrarily larger than E_d.) For $T < \Delta$, χ will be of the order of the inverse of the crossover temperature (i.e., $\chi \approx \Delta^{-1}$ as predicted by Varma and Yafet[2]). Since this is a crossover region between two simple limits ("integral valence"), low-temperature properties will be sensitive—though universal—functions of E_d^*/Δ.

When $-E_d^* \gg \Delta$ [Fig. 2(c)], renormalization stops when the state $|0\rangle$ is decoupled at $W \approx -E_d(W) \approx \tilde{T}$ ($\gg \Delta$), where

$$\tilde{T} - (\Delta/\pi)\ln(\bar{\alpha}\tilde{T}/\Delta) = -E_d^*; \tag{16}$$

$\bar{\alpha}$ is analogous to α in (14).[10] For $T < \tilde{T}$, charge fluctuations are frozen out leaving $\langle n_d\rangle \simeq 1$, and a local moment. The Schrieffer-Wolff transformation[11] to a Kondo model is then valid, giving $(J\rho)^{\rm eff} = -2\Delta/\pi\tilde{T}$, $D^{\rm eff} \approx \tilde{T}$. Below a Kondo temperature $T_{\rm K}$, given by[5] $D(|J\rho|)^{1/2}\exp(1/J\rho)$, the local moment is quenched, leaving a Fermi liquid. From (16), $T_{\rm K}$ is of order $\exp(\pi E_d^*/2\Delta)$, hence, $T_{\rm K} \ll \Delta$; in terms of bare parameters, $T_{\rm K} \approx (\omega_0\Delta)^{1/2}\exp(\pi E_d/2\Delta)$, in agreement with a form obtained recently by perturbation theory.[12]

An analytic scaling theory is able to identify universality and scaling invariants, but only qualitatively describes the crossovers; this is where Wilson's numerical technique[5,3] comes into its own. The main candidate for a detailed numerical study is the low-temperature mixed-valence region, where, when energies are expressed in units of Δ, the susceptibility, linear specific heat coefficient, and $\langle n_d\rangle$ should smoothly change from their Kondo to $E_d^* \gg \Delta$ values as universal functions of E_d^*/Δ.

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[8]W_0 may be precisely defined by the value W in the perturbation expansion (2).

[9]Notionally, "scaling laws" may be derived in the limits $\omega \ll |E_d|$ or $\omega \ll T$, provided $\omega \gg \Delta$, by using a similar procedure to (5)–(12). The resulting equations are "trivial" in the manner of (12), as states in this energy range do not cause any renormalization of E_d or Δ. In the crossover regions $\omega \approx |E_d|$ or $\omega \approx T$, such "scaling equations" are not well defined, as they depend on the cutoff structure.

[10]α and $\bar{\alpha}$ relate $E_d(T=0)$ ($= T^*$ or $-\tilde{T}$) to $E_d(T)$ at temperatures above the crossover. They may be obtained by detailed comparison of the perturbation expansions when T is above and below the crossover {F. D. M. Haldane, to be published; comparison of perturbation theory in Δ for the Anderson model with $W = \infty$ and $E_d + U$, $-E_d \gg T$, with that for the Kondo model shows that for $U \gg \Delta$, $T_{\rm K}$ is proportional to $(U\Delta)^{1/2}\exp[E_d(E_d+U)/(2\Delta U/\pi)]$}. In fact, $\alpha = \bar{\alpha}$.

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