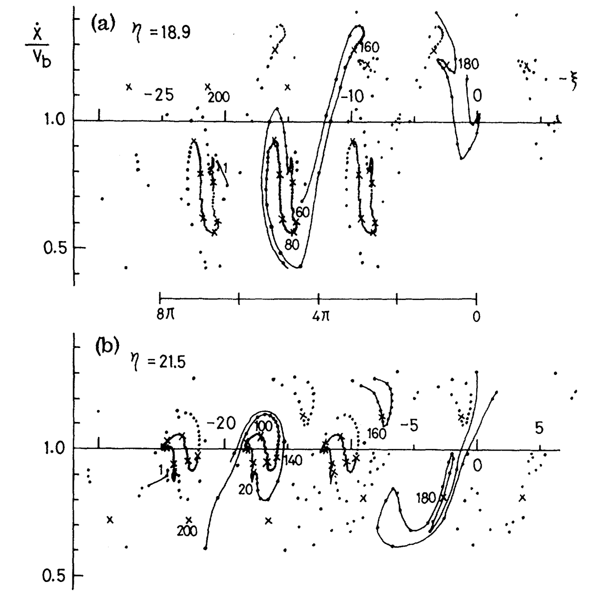

FIG. 4. Phase-space loci in the collisionless case plotted for two positions, viz., (a) the third maximum and (b) the third minimum of $E^2(\eta)$, respectively. Other conditions are the same as in Fig. 3.

damping of the amplitude near η_2. As is shown in Fig. 3, after η_2, the beam electrons spill into adjacent wave troughs and spread in phase space. This tendency becomes even greater beyond η_2. Furthermore, when we compared phase-space trajectories of some test particles in the $\dot{x}$-η plane in the collisional case with ones in the collisionless case, we found that they clearly departed from each other between η_3 and η_4. Motions of test particles in the collisional case become more irregular and their bounce periods also become longer because of the reduced amplitude of the wave. The energy exchange between the wave and the beam electrons ceases after η_3 as shown in Fig. 1.

Oscillations could be destroyed[3] as a result of particle phase mixing by either (a) wave damping or (b) modulation of the main wave by unstable sidebands. In the former case, a catastrophic effect on the oscillation is expected, similar to the results presented here.

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Shear, Periodicity, and Plasma Ballooning Modes

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A procedure which reconciles long parallel wavelength, characteristic of plasma instabilities, with periodicity in a sheared toroidal magnetic field is described. Applied to the problem of high-n ballooning modes in tokamaks it makes possible a full minimization of the potential energy functional δW and shows that previous calculations overestimated stability.

In many investigations of plasma stability,[1-3] in both fluid and kinetic theories, the principal difficulty is that of reconciling the characteristics of unstable oscillations—such as long parallel wavelength and short perpendicular wavelength—with the constraints imposed by periodicity in a sheared toroidal magnetic field. In this Letter we describe a general method for overcoming this difficulty and apply it to the important problem[1,2] of determining the stability limit for ballooning modes—which in turn determines the maximum β attainable in a tokamak.

To describe an axisymmetric toroidal system we may choose a set of orthogonal coordinates[4]

(ψ, χ, ζ) in which ψ labels the magnetic surfaces, χ is a poloidal anglelike variable, and ζ is the toroidal angle. The magnetic field is $\vec{B} = \nabla\psi\times\nabla\zeta + f(\psi)\nabla\zeta$ and the metric is

$$(ds)^2 = (d\psi/RB_p)^2 + (JB_p\,d\chi)^2 + (Rd\zeta)^2,$$

with J the Jacobian, R the radius, and B_p the poloidal field $|\nabla\psi\times\nabla\zeta|$. We also define $\nu = fJ/R^2$ so that $\oint \nu d\chi = 2\pi q$ where q is the "safety factor."

Then the conventional representation of waves with short perpendicular and long parallel wavelength would be in the eikonal form

$$\varphi(\psi, \chi)\exp(in[\zeta - \int^{\chi}\nu d\chi]), \tag{1}$$

with $n \gg 1$ and φ varying slowly compared to the phase. (The phase is constant along $\vec{B}$ but varies rapidly perpendicular to $\vec{B}$.) However it is easily seen that in a magnetic field with shear $dq/d\psi \neq 0$, this form is incompatible with periodicity in χ.

Recent calculations of ballooning modes[1,2] attempted to overcome this difficulty by imposing an artificial constraint that $\varphi = 0$ at the ends of the basic interval in χ. However, as we shall show, one does not then obtain the most unstable mode so that such calculations overestimate the stability of the system. Other authors[3] have attempted to circumvent the problem by introducing discontinuous jumps in φ at the ends of the basic interval in χ but this is incompatible with the assumption that the amplitude $\varphi(\psi, \chi)$ varies slowly compared to the phase. An alternative approach[5,2] is to modify the eikonal by an arbitrary function G such that $\oint(\nu + G)d\chi = 2\pi m$ on all surfaces but no satisfactory method for determining G has been given.

These difficulties can be overcome as follows.[6] In any axisymmetric system the determination of stability can be reduced to a two-dimensional eigenvalue problem

$$\mathcal{L}(\theta, x)\varphi(\theta, x) = \lambda\varphi(\theta, x), \tag{2}$$

where θ represents the poloidal angle and x the flux surface coordinate. The operator $\mathcal{L}$ is periodic in θ and $\varphi(\theta, x)$ must also be periodic in θ and bounded in x. We now express φ in the form

$$\varphi(\theta, x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}\hat{\varphi}(\eta, x)\,d\eta. \tag{3}$$

This transformation can be regarded as the following sequence: (i) representation of the periodic function φ by a Fourier sum with coefficients a_m, (ii) analytic continuation of a_m to a function $a(m)$, and (iii) representation of $a(m)$ as a Fourier integral. Any $\hat{\varphi}(\eta, x)$ which is a solution of the equation

$$\mathcal{L}(\eta, x)\hat{\varphi}(\eta, x) = \lambda\hat{\varphi}(\eta, x) \tag{4}$$

in the *infinite* domain, $-\infty < \eta < \infty$, will generate a periodic solution of Eq. (2) with the same eigenvalue and it can be shown that there are no other solutions of Eq. (2). (A fuller discussion will be given in a later paper.)

The function $\hat{\varphi}$ *need not* be periodic and *can* therefore be expressed in the eikonal form (1). Of course $\hat{\varphi}(\eta, x)$ is not the actual plasma perturbation, but the real, periodic, perturbation can be constructed from it.

We now apply this technique to the problem of high-mode-number ballooning modes[1,2] in tokamaks. Stability of these modes can be determined by minimizing the potential energy functional $\delta W(\xi, \xi)$. The perturbation is decomposed into modes $\sim \exp in\zeta$ and, provided shear is nonvanishing, δW is minimized by displacements which are divergence free; then δW can be expressed in terms of the components of ξ perpendicular to $\vec{B}$. For small values of n further minimization of δW has been done numerically,[7,8] but this fails for large n. However in this limit the minimization can be performed analytically. When $n \gg 1$, δW will be positive unless the perpendicular gradients of the perturbation are of order n but the parallel gradients remain of order unity (i.e., unless the mode varies rapidly perpendicular to $\vec{B}$ but slowly parallel to $\vec{B}$). The divergence $(\nabla\cdot\vec{\xi}_\perp)$ must also be of order unity. A further minimization can be carried out and δW then depends only on the normal displacement through $X \equiv RB_p\,\xi_\psi$. Thus, in an expansion in $1/n$, the dominant contribution to δW is

$$\delta W_0 = \pi\int Jd\chi\,d\psi\left[\frac{B^2}{R^2B_p^{\,2}}|k_\parallel X|^2 + R^2B_p^{\,2}\left|\frac{1}{n}\frac{\partial}{\partial\psi}k_\parallel X\right|^2 - 2\frac{dp}{d\psi}\left(\frac{\kappa_n}{RB_p}|X|^2 - \frac{ifB_p}{B^2}\kappa_s\frac{X}{n}\frac{\partial X^*}{\partial\psi}\right)\right], \tag{5}$$

where κ_n and κ_s are the normal and geodesic components of the curvature $\vec{\kappa} = -\vec{B}\times[\vec{B}\times\nabla(p + \frac{1}{2}B^2)]\,B^{-4}$ and

$$ik_\parallel \equiv (JB)^{-1}(\partial/\partial\chi + in\nu).$$

The Euler equation, obtained by minimizing (5) over all functions $X(\psi,\chi)$ which are periodic in χ, is

$$Bk_{\parallel}\left\{\frac{B}{R^2B_p^2}\left[1-\left(\frac{R^2B_p^2}{nB}\right)^2\frac{\partial^2}{\partial\psi^2}\right]k_{\parallel}X\right\}-2\frac{dp}{d\psi}\left(\frac{\kappa_n X}{RB_p}-\frac{i\kappa_s fB_p}{B^2 n}\frac{\partial X}{\partial\psi}\right)=0. \tag{6}$$

Thus far, our calculation follows that of Ref. 1. However we now make the crucial step by introducing the transformation discussed earlier, namely

$$X(\psi,\chi)=\sum_m \exp\left(\frac{-2\pi im\chi}{\oint d\chi}\right)\int_{-\infty}^{+\infty}\exp\left(\frac{2\pi imy}{\oint d\chi}\right)\hat{X}(\psi,y)\,dy. \tag{7}$$

Then $X(\psi,\chi)$ will be periodic in χ and will satisfy the Euler equation (6) provided that $\hat{X}$ satisfies the same equation in the infinite domain $-\infty<y<\infty$. The solution of Eq. (6) in the infinite domain may be obtained by writing $\hat{X}$ in the form of a (nonperiodic) "quasimode"[9]

$$\hat{X}(\psi,y)=F(\psi,y)\exp(-in\textstyle\int^y \nu\,dy), \tag{8}$$

where the exponential factor contains all the rapid cross-field variation and where F satisfies the ordinary differential equation

$$\frac{1}{J}\frac{d}{dy}\left\{\frac{1}{JR^2B_p^2}\left[1+\left(\frac{R^2B_p^2}{B}\int^y\frac{\partial\nu}{\partial\psi}dy\right)^2\right]\frac{dF}{dy}\right\}+\frac{2}{RB_p}\frac{dp}{d\psi}\left(\kappa_n-\frac{fRB_p^2}{B^2}\kappa_s\int^y\frac{\partial\nu}{\partial\psi}dy\right)F=0 \tag{9}$$

in which ψ appears only as a parameter. Equation (9) can readily be solved for any prescribed equilibrium and determines its stability against high-n ballooning modes. (In this lowest-order calculation the slow ψ variation of F is not determined; it can be obtained from higher orders of the $1/n$ expansion.[6])

We see, therefore, that the introduction of the transformation (7) *followed by* the quasimode form (8) decouples the stability analysis from surface to surface and provides a complete minimization of δW at large n. {If the quasimode form were introduced directly for $X(\psi,\chi)$, as in Ref. 1, one would obtain Eq. (9) [which is identical to Eq. (6) of Ref. 1] but with χ as the independent variable. However, since the quasimode form is not periodic, an additional constraint such as $F(\chi)=0$ at $\chi=\pm\frac{1}{2}\oint d\chi$ would be required to ensure periodicity and one would not then obtain the full minimization of δW.}

The correct boundary conditions on $F(y)$ are obtained by considering the behavior of Eq. (9) as $|y|\to\infty$. In this limit $F\sim F_0(y)+y^{-1}F_1(y)$, where $F_0(y)$ is independent of the periodic poloidal variation of the equilibrium and is a solution of the averaged equation

$$\frac{d}{dy}y^2\frac{dF_0}{dy}+DF_0=0,$$

where the coefficient D is that which appears in Mercier's necessary criterion for stability,[4] $D<\frac{1}{4}$. Thus the asymptotic solution is $F\sim(Ay^{\alpha_1}+By^{\alpha_2})$, where $\alpha_{1,2}=-\frac{1}{2}\pm(\frac{1}{4}-D)^{1/2}$. If $D>\frac{1}{4}$ both asymptotic solutions are acceptable and an unstable displacement can always be found. (Hence our analysis incorporates the Mercier criterion.) If $D<\frac{1}{4}$ only the small solution is acceptable (the large solution leads to a divergent δW) and this provides the appropriate boundary condition for Eq. (9).

As a specific example we have considered a model problem representing a large-aspect-ratio tokamak with circular flux surfaces. In this model the magnetic field is uniform over the magnetic surface but the shear is nonuniform. Equation (9) becomes

$$\frac{d}{d\eta}[1+(s\eta-\alpha\sin\eta)^2]\frac{dF}{d\eta}+\alpha[\cos\eta+\sin\eta(s\eta-\alpha\sin\eta)]F=0, \tag{10}$$

where

$$s\equiv d(\ln q)/d(\ln r) \text{ and } \alpha\equiv-(2Rq^2/B^2)\,dp/dr$$

are measures of the mean shear and pressure gradient, respectively.

Equation (10) has been integrated numerically with the boundary condition $F\to 0$ as $|\eta|\to\infty$ (which is numerically equivalent to selecting only the small asymptotic solution). The boundary between stability and instability is shown in Fig. 1, which indicates that, for this model, the critical pressure gradient for ballooning modes is rather insensitive to shear. Over most of the range it is roughly $dp/dr\sim 0.25\,B_0^2/Rq^2$. Also shown in Fig. 1 (dotted line) is the stability boundary obtained by imposing the boundary condition $F(\pm\pi)=0$ used by Dobrott *et al.*[1] This overestimates

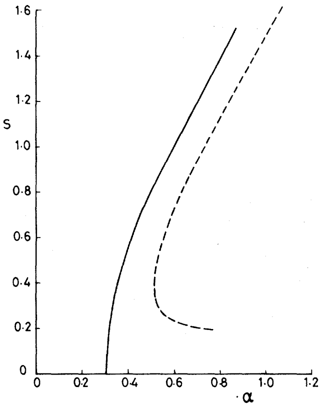

FIG. 1. Maximum stable pressure gradient α as function of shear s.

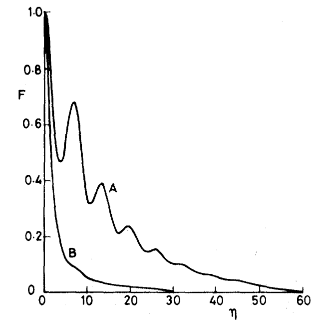

FIG. 2. Marginally stable eigenfunctions. Curve A, low shear $s = 0.1$; curve B, high shear $s = 0.7$.

the stability of the system and produces a threshold value of dp/dr about 20% greater. This effect becomes more marked at low shear because, as shown in Fig. 2, the eigenfunction $F(\eta)$ then extends considerably beyond $\pm\pi$.

In conclusion, we have shown that a complete minimization of δW in the limit $n \to \infty$ can be obtained by the transformation (7) together with the quasimode form (8). This reduces the problem to an ordinary differential equation which can readily be solved to determine the stability of any prescribed equilibrium and whose asymptotic behavior is related to the Mercier criterion.[10]

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