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Integration of Linearized Evolution Equations

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This paper treats the problem of solving the linear equation which arises when a nonlinear evolution equation is linearized around some particular solution. It is shown that if the original equation is of completely integrable Hamiltonian form there are an infinity of explicit solutions of the linearized equation. These are almost always linearly independent.

Frequently, one encounters the following problem: We have some nonlinear evolution equation and a particular solution of it. It is desired, for example, to study stability or quantization, to discuss the equation linearized around the particular solution. Now it is well known that if the original equation is translationally invariant one can immediately write down a solution of the linearized equation. Thus, if the original equation is invariant under spatial translation and u is a solution, then $\partial u/\partial x$ is a solution of the linearized equation. Similarly, if we have time translation invariance, $\partial u/\partial t$ is a solution. In general these two solutions are linearly independent and nontrivial. There is one special case. If $u = u(x-ct)$, then these solutions are proportional.

Here we wish to point out that for a very large class of evolutions many more explicit solutions of the linearized equations are readily obtained. These solutions are related to the densities of conserved functionals. The class of evolution equations involved appears to include *all* of those which are known to be completely integrable by the inverse-scattering transform method and even some for which this is not known. Since in the former case we know that there are an infinity of conserved functionals, we obtain an infinity of explicit solutions of the linearized equations.

The essential theorem[1] used to obtain our results is the following: Consider a Hamiltonian system. We have a Hamiltonian H and an appropriately defined Poisson bracket $[\,,\,]$ such that the equations of motion are of the form

$$dF/dt = [F, H]. \tag{1}$$

Let G be a conserved functional, i.e.,

$$[G, H] = 0; \tag{2}$$

then a solution of Eq. (1) linearized around a particular solution $F^{(0)}$ is

$$F^{(1)} = [F^{(0)}, G]. \tag{3}$$

In words, the infinitesimal contact transformation generated by G yields a solution of the linearized equation.

We consider here two applications.

(1) *The situation envisaged by Lax.*[2]—Here the evolution equation is of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)}, \tag{4}$$

where H is some functional of u. Poisson brackets between two functionals F_1, F_2 are defined by

$$[F_1, F_2] = \int \frac{\delta F_1}{\delta u(x')}\frac{\partial}{\partial x'}\frac{\delta F_2}{\delta u(x')}\,dx'.$$

Then Eq. (4) is of Hamiltonian form,

$$\partial u/\partial t = [u, H]; \tag{5}$$

and conserved functionals, i.e., such that

$$dF/dt=0, \tag{6}$$

satisfy

$$\int \frac{\delta F}{\delta u}\frac{\partial u}{\partial t}=\int \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u}\,dx=[F,H]=0. \tag{7}$$

Now Eq. (4) linearized around some solution u has the form

$$\partial v/\partial t=(\partial/\partial x)(Nv), \tag{8}$$

where the *linear* operator N is defined by

$$Nv=\frac{d}{d\epsilon}\frac{\delta H(u+\epsilon v)}{\delta u}\bigg|_{\epsilon=0}. \tag{9}$$

The theorem tells us that if $F[u]$ is a conserved functional, we have as a solution of Eq. (8)

$$\begin{aligned} v(x,t)=[u,F]&=\int \frac{\delta u(x')}{\delta u(x)}\frac{\partial}{\partial x'}\frac{\delta F}{\delta u(x')}\,dx \\ &=\int \delta(x-x')\frac{\partial}{\partial x'}\frac{\delta F}{\delta u(x')}\,dx' \\ &=\frac{\partial}{\partial x}\frac{\delta F}{\delta u(x)}. \end{aligned} \tag{10}$$

Thus, the derivative of the conserved density satisfies the linearized equation.

There are three "classical" conserved functionals corresponding to Eq. (4). These are

$$I_1=\int u\,dx,\quad I_2=\int \tfrac{1}{2}u^2\,dx,\quad I_3=H,$$

with functional derivatives $\varphi_1=1$, $\varphi_2=u$, $\varphi_3=\delta H/\delta u$. The corresponding solutions of Eq. (8) are

$$\partial\varphi_1/\partial x=0,$$

which is trivial, and

$$\frac{\partial\varphi_2}{\partial x}=\frac{\partial u}{\partial x}, \tag{11}$$

$$\frac{\partial\varphi_3}{\partial x}=\frac{\partial}{\partial x}\frac{\delta H}{\delta u}=\frac{\partial u}{\partial t}, \tag{12}$$

which are the solutions mentioned in the beginning.

However, for special forms of H there may be more conserved functionals. For example, a typical equation of the form of Eq. (4) is

$$\frac{\partial u}{\partial t}=u^P\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}G(x'-x)\frac{\partial u}{\partial x'}\,dx', \tag{13}$$

with $G(x)=-G(-x)$. Here,

$$H=\int_{-\infty}^{\infty}\frac{u^{P+2}\,dx}{(p+1)(p+2)} + \frac{1}{2}\iint_{-\infty}^{\infty}\frac{\partial u}{\partial x}\hat{G}(x'-x)\frac{\partial u}{\partial x'}\,dx\,dx', \tag{14}$$

with $\partial\hat{G}/\partial x=G(x)$. Thus

$$\frac{\delta H}{\delta u}=\frac{u^{P+1}}{p+1}+\int_{-\infty}^{\infty}G(x'-x)\frac{\partial u}{\partial x'}\,dx'. \tag{15}$$

The linearized form of Eq. (13) is then

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left\{u^P v+\int_{-\infty}^{\infty}G(x'-x)\frac{\partial v}{\partial x'}\,dx'\right\}. \tag{16}$$

If $p=1$, $G(x)=-\delta'(x)$, then Eq. (13) becomes the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}=u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}. \tag{17}$$

This, it is known, has an infinite number of conserved functionals. Thus, we have an infinite number of solutions of the specialization of Eq. (16),[3]

$$\frac{\partial v}{\partial t}=\frac{\partial u}{\partial x}v+\frac{\partial v}{\partial x}u+\frac{\partial^3 v}{\partial x^3}. \tag{18}$$

The simplest nonclassical of these corresponds to the conserved functional

$$I_4=\int_{-\infty}^{\infty}\left\{\frac{u^4}{12}-u\left(\frac{\partial u}{\partial x}\right)^2+\frac{3}{5}\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right\}dx. \tag{19}$$

Thus,

$$\varphi_4\equiv\frac{\delta I_4}{\delta u}=\frac{u^3}{3}+\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}+\frac{6}{5}\frac{\partial^4 u}{\partial x^4}. \tag{20}$$

In addition to solutions $v=u_t$, $v=u_x$ we then have the solution

$$v=\partial\varphi_4/\partial x. \tag{21}$$

If $p=1$, $G=1/x$,[4] then Eq. (13) becomes the Benjamin-Ono equation

$$\frac{\partial u}{\partial t}=u\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{\partial u/\partial x'}{x'-x}\,dx'. \tag{22}$$

This is known[5,6] to have at least one more conserved functional than the classical ones. It is

$$I_4=\int_{-\infty}^{\infty}\left[\frac{u^4}{12}+\frac{2\pi^2}{3}\left(\frac{\partial u}{\partial x}\right)^2\right]dx + \iint_{-\infty}^{\infty}\frac{u(x)u(x')\partial u/\partial x'}{x'-x}\,dx\,dx'. \tag{23}$$

From this we conclude that if u is a solution of

Eq. (22), then $v = \partial\varphi_4/\partial x$ is a solution of

$$\frac{\partial v}{\partial t} = u_x v + u v_x + \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \frac{\partial v/\partial x'}{x'-x}dx', \tag{24}$$

where

$$\varphi_4 = \frac{u^3}{3} - 3\pi^2\frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{\infty}\frac{[u(x)+u(x')]}{x'-x}\frac{\partial u}{\partial x'}dx'. \tag{25}$$

We should consider the question of linear independence[7] of the solutions obtained. These all seem to be independent except for one very special (but unfortunately very interesting) case. That is the case where $u = u(x-ct)$—the single-soliton case. As we saw, the solutions $v = u_x$ and $v = u_t$ are proportional. The case of the general conserved functional is readily discussed in the Korteweg–de Vries case. Then,[8] the density of the nth conserved polynomial functional can be written in the form

$$\varphi_n = M^{n-1}1, \tag{26}$$

where the operator M is defined by

$$M\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{u\psi}{3} + \frac{1}{3}\int_{-\infty}^{x} u(x')\frac{\partial\psi}{\partial x'}dx'. \tag{27}$$

Thus, $\varphi_1 = 1$, $\varphi_2 = u/3$, and

$$\varphi_3 = \frac{1}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{u^2}{2}\right). \tag{28}$$

However, if in Eq. (17) we put $u = u(x-ct)$ and integrate, we obtain

$$-cu = \frac{\partial^2 u}{\partial x^2} + \frac{u^2}{2} \tag{29}$$

Thus,

$$\varphi_3 = -cu/3 \tag{30}$$

and, in general,

$$\varphi_n = (-c)^{n-2}u/3, \quad n \geq 2. \tag{31}$$

The reason for this degeneracy appears to be that for solutions of this special form the time derivative is equivalent to the space derivative. In other cases the different conserved functionals give rise to different solutions of the linearized equation. For example, in the n-soliton case we know that asymptotically u has the form

$$u(x,t) = \textstyle\sum_i u_i(x-c_i t), \quad c_i \neq c_j \text{ for } i \neq j. \tag{32}$$

Then certainly $u_t \neq u_x$.

(2) *The Toda lattice.*—The equations as originally formulated[9] are already in Hamiltonian form. They are

$$\dot{q}_n = p_n,$$
$$\dot{p}_n = -\{\exp[-(q_{n+1}-q_n)] - \exp[-(q_n - q_{n-1})]\}. \tag{33}$$

Flaschka[10] has shown the existence of an infinite number of conserved functionals F_N. Using these, we construct solutions of the equations linearized around a given solution by

$$\begin{aligned} q_n^{(1)} &= [q_n^{(0)}, F_N(q^{(0)}, p^{(0)})] = \partial F_N/\partial p_n^{(0)}, \\ p_n^{(1)} &= [p_n^{(0)}, F_N(q^{(0)}, p^{(0)})] = \partial F_N/\partial q_n^{(0)}. \end{aligned} \tag{34}$$

The first three nontrivial constants are

$$F_2 = \sum_{m=-\infty}^{\infty} p_m, \tag{35}$$

$$H \sim F_3 = \sum_{m=-\infty}^{\infty}\{\exp[-(q_m - q_{m-1})] - 1\} + p_m^2/2, \tag{36}$$

$$F_4 = \sum_{m=-\infty}^{\infty}\{\exp[-(q_m - q_{m-1})]\}[p_{m-1} + p_{m-2}] + (p_m)^3. \tag{37}$$

Corresponding to the two "classical" constants F_2 and F_3, we have the solutions

$$q_n^{(1)} = 1, \quad p_n^{(1)} = 0 \text{ and } q_n^{(1)} = \dot{q}_n^{(0)}, \quad p_n^{(1)} = \dot{p}_n^{(0)}, \tag{38}$$

while for the nonclassical constant F_4, we obtain

$$q_n^{(1)} = \{\exp[-(q_{n+1}^{(0)} - q_n^{(0)})] + \exp[-(q_{n+2}^{(0)} - q_{n+1}^{(0)})]\} + 3(p_n^{(0)})^2, \tag{39}$$

$$p_n^{(1)} = \{\exp[-(q_n^{(0)} - q_{n-1}^{(0)})]\}[p_{n-1}^{(0)} + p_{n-2}^{(0)}] - \{\exp[-(q_{n+1}^{(0)} - q_n^{(0)})]\}[p_n^{(0)} + p_{n-1}^{(0)}]. \tag{40}$$

Contrary to the continuous case discussed above even the solutions obtained around the one-soliton solution are linearly independent.

[1]See, for example, E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge Univ. Press, Cambridge, England, 1937), 4th Ed., Sect. 144.

[2]Peter D. Lax, Commun. Pure Appl. Math. 28, 141 (1975).

[3]Clearly, this implies a high-degree stability of the solutions of the Korteweg–de Vries equation. The solutions obtained for the linearized equation are only polynomials of the u and its derivatives. (If these solutions are complete, which is not yet known, this proves general linear stability.) In any event, this proves stability under the large class of perturbations which can be approximated by linear combinations of our linear solutions.

[4]By $1/x$, we mean the principal-value interpretation.

[5]K. M. Case, Stanford Research Institute Report No. JSS-77-31 (unpublished); J. D. Meiss and N. R. Pereira, to be published.

[6]The Benjamin-Ono equation has a two-fold interest: (a) physically it describes a large class of internal waves which occur both in the atmosphere and the ocean; (b) mathematically it is of interest in that while it is not known whether it is completely integrable or not, it does share many properties with equations which are. Thus, (i) it does have at least one nonclassical constant, and (ii) it has solutions of N-soliton type for arbitrary N.

[7]By linear independence we mean, of course, that there is no nontrivial linear combination of the solutions with constant coefficients which is identically zero.

[8]H. H. Chen, Y. C. Lee, and C. S. Liu, to be published.

[9]M. Toda, Prog. Theor. Phys. Suppl. 45, 174 (1970).

[10]H. Flaschka, Prog. Theor. Phys. 51, 703 (1974).