modes of different n becomes more pronounced as $k_y\rho_s \to 1$, and the eigenmodes become nearly degenerate and difficult to differentiate. We have also carried out numerical studies with finite ion temperature, and electron and ion temperature gradients, employing the complete collisionless slab model of Ref. 2. For $L_n/L_s = 0.02$ and $0 \leq T_i/T_e \leq 1$, no growing modes were found for $-1 \leq \eta_e, \eta_i \leq 1$, where $\eta_j = d\ln T_j/d\ln n$. For $k_y\rho_s \to 1$, the damping rates remained near zero over the entire range of parameters. For $k_y\rho_s \ll 1$, the effect of a reversed electron temperature gradient, $\eta_e < 0$, was found to be in the direction of instability, as expected, but unable to overcome the shear stabilization.

Finally, we remark that we have not shown all drift waves to be stable. In a torus, there are a number of effects, such as ballooning or trapped-particle collisions and drifts, which can lead to instability. Further, the existence of convectively growing wave packets should be reexamined in the light of the present work.

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"Absolute Universal Instability" Is Not Universal

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The roots of an improved analytic eigenvalue equation for the absolute universal or collisionless drift instability in a sheared magnetic field are found numerically and compared with the eigenvalues obtained from a numerical solution of the exact differential equation. The startling result is that both techniques predict stability, no matter how weak the shear or how large the transverse wave number, in contradiction to all previous work. Stability is due primarily to the stabilizing influence of the nonresonant electrons.

Since the pioneering work of Pearlstein and Berk,[1] the instability of the collisionless drift wave in a sheared magnetic field has been the subject of numerous investigations.[2-4] This instability, which is driven by the the wave-particle interaction between the drift wave and the electrons, is also termed the "universal instability" because its existence requires only a density gradient which is a necessary feature of a confined plasma. Previous investigations have employed perturbation theory[2] or approximate numerical solutions[3] of the perturbation-theory solution near marginal stability.[4] In Ref. 4 the same differential equation is solved by breaking up the spatial domain into inner and outer regions. In the outer region, the resonant electron term is subdominant and the equation can be solved iteratively. This outer solution is then matched to a jump condition derived in the inner region.

Recently, Catto and Tsang[5] extended the work of Rosenbluth and Catto[4] to obtain an improved eigenvalue equation for all even and odd radial eigenmodes. More importantly, they were able to discover the limit in which the perturbation-theory results could be recovered from the more exact expressions valid for arbitrary growth rates.[5,6] As a result, there emerged the possibility that the perturbation-theory form of the dispersion relation is inadequate because it can only be recovered in a limit in which small corrections can be important. In particular, the per-

turbation-theory form was found to be more accurate for the more strongly damped modes.[5] For growing and marginally stable modes the perturbation theory is expected to be only qualitatively correct.

In this Letter, we compare the improved analytic result of Refs. 4 and 5 with the perturbation form and with a numerical solution of the relevant differential equation. The agreement between the improved analytic result and the numerical solution is found to be remarkably good. To our surprise, in contrast to the perturbation form, both the analytic result and the numerical solution show no instability at all for a wide range of parameters. Our results[6] to date indicate that the collisionless drift wave is always stable for the parameters of interest in tokamaks.

It seems likely that all previous work has been too strongly influenced by the perturbation-theory form of the eigenvalue equation. In fairness to previous workers, it must be pointed out that all three results have qualitatively the same behavior in paramater space. However, only the perturbation-theory form of the eigenvalue equation consistently predicts instability for parameters relevant to the tokamak.

In addition to the misconception based on the perturbation theory, it is also commonly believed that the electron Z function that describes the electron response is well approximated by its residue. When this approximation is employed in the numerical solution of the differential equation, however, instability is predicted. Near the rational surface, the electron Z function varies rapidly and is poorly represented by its residue. In this region, reliable results can be obtained only by a numerical scheme with a sufficiently small grid.

Furthermore, away from the rational surface where the residue becomes an accurate approximation, a comparison of the perturbation theory with improved analytic results indicates that the wave-electron interaction is somewhat less destabilizing than is normally believed. This conclusion follows because the perturbative treatment employs a lowest-order eigenfunction that neglects the stabilizing deformations due to the radial structure of the resonant term; thus, it cannot properly predict the stability regions for the collisionless drift wave. The improved analytic model retains these corrections.

In this Letter, effects due to temperature gradients are not retained. However, they are expected to be stabilizing for normal profiles. Furthermore, only the most unstable (lowest even) eigenmode is considered, although the next mode (lowest odd) has been considered also and found to be more stable as expected.

In the absence of temperature gradients, the eigenmode equation for the collisionless drift instability is of the form[5]

$$\frac{\partial^2\Phi}{\partial x^2}-\left[\Lambda-\mu^2x^2+\frac{\sigma(|x|)}{|x|}\right]\Phi=1, \tag{1}$$

where the electrostatic potential is written as $\Phi(x)\exp(-i\omega t+iky)$. In Eq. (1), Λ contains the basic drift-wave terms,

$$\Lambda=\rho_i{}^{-2}D^{-1}\{\omega[1+\tau(1-\Gamma_0)]-\omega_*\Gamma_0\},$$

with $D=(\omega\tau+\omega_*)(\Gamma_0-\Gamma_1)$, where $\rho_i=(c/eB)\times(M_iT_i)^{1/2}$ is the ion gyroradius, $\omega_*=kcT_e/(eB/L_n)$ is the electron diamagnetic drift, $\tau=T_e/T_i$ is the temperature ratio, $\Gamma_n=I_n(b)\exp(-b)$, I_n is the modified Bessel function, and $b=(k\rho_i)^2$. The μ^2x^2 term is the usual shear term due to the ion inertia, and μ is defined by

$$\mu=(\omega_*L_n/\omega\tau L_s\rho_i{}^2)S^{1/2},$$

where

$$S=D^{-1}(\omega\tau+\omega_*)\Gamma_0, \quad L_n{}^{-1}=-\partial\ln N/\partial x,$$

N is the density, and L_s is the shear length. The $\partial^2\Phi/\partial x^2$ term appears because of the finite gyration radius of the ions; σ/x is the destabilizing resonant electron contribution, with σ defined as

$$\sigma(|x|)=\rho_i{}^{-2}D^{-1}(\omega-\omega_*)x_eZ(x_e/|x|),$$

where $x_e=\omega L_s/kv_e$ and $v_e=(2T_e/M_e)^{1/2}$ is the electron thermal velocity.

Equation (1) can be solved by the method of matched asymptotic expansions.[4,5] In the outer region, the σ term is treated iteratively, while in the inner region, it is the dominant term in the coefficient of Φ. The boundary conditions are that Φ must be outgoing at infinity and either Φ or Φ' vanishes at $x=0$ depending on whether Φ is odd or even. The eigenvalue equation for all the even radial modes is

$$1-\frac{\sigma_0}{2\pi(i\mu)^{1/2}}\Gamma\left(\frac{1}{4}-\frac{i\Lambda}{4\mu}\right)\Gamma\left(\frac{1}{4}+\frac{i\Lambda}{4\mu}\right)\sin\left[\frac{\pi}{4}\left(1+\frac{i\Lambda}{4\mu}\right)\right]\ln[x_e(2\mu)^{1/2}]=0, \tag{2}$$

where

$$\sigma_0 = i\pi^{1/2} x_e \rho_i^{-2} D^{-1}(\omega - \omega_*).$$

Equation (2) can be obtained from the lowest-order form of Eq. (25) in Ref. 4. In Ref. 5, it is pointed out that the perturbation-theory form of the eigenvalue equation can be recovered by employing the representation for the Γ function valid in the vicinity of the poles (argument equal to zero or a negative integer). For the lowest even mode, the perturbation theory form is recovered by replacing $\Gamma(\frac{1}{4} - i\Lambda/4\mu)$ by its small-argument limit and employing $\Lambda \approx -i\mu$ elsewhere to obtain

$$\Lambda + i\mu = 2\sigma_0 (i\mu/\pi)^{1/2} \ln[x_e (2\mu)^{1/2}]. \tag{3}$$

Rather than determine the numerical solution of Eq. (1), we retain the full ion Z function so that the eigenmode equation to be solved numerically is

$$\rho_i^2 (\Gamma - \Gamma_1) \frac{x_i}{x} Z\left(\frac{x_i}{x}\right) \frac{\partial^2 \Phi}{\partial x^2} + \left\{ \frac{\omega - \omega_*}{\tau\omega + \omega_*} \left[1 + \frac{x_e}{x} Z\left(\frac{x_e}{x}\right) \right] + 1 + \frac{x_i}{x} Z\left(\frac{x_i}{x}\right) \Gamma_0 \right\} \Phi = 0, \tag{4}$$

where $x_i = \omega L_s / k v_i$ and $v_i = (2T_i/M_i)^{1/2}$ is the ion thermal velocity. The lowest even mode of Eq. (4) is found numerically by an invariant imbedding method.[7] The boundary condition at infinity is satisfied by picking $\Phi = 0$ at $x = l$ and shooting inward. The shooting length l varies in practice about $10\rho_i$ to $75\rho_i$, depending on whether or not the mode is strongly spatially damped. Use of the WKB solution to first order instead of $\Phi = 0$ to start the integration permits shorter shooting lengths. The results of both methods agree if l is sufficiently large. However, care must always be taken to verify that the eigenvalue is insensitive to the shooting length. A variable-step integrator[8] is used to handle the fast x variation of $Z(x_e/x)$. The complex secant method is employed to locate the value of ω such that $\Phi'(x = 0, \omega) = 0$. Eigenfunctions are easily recovered.

In Fig. 1, the imaginary part of the eigenfrequency associated with the improved analytic eigenvalue equation [Eq. (2)] is compared with that obtained from the numerical solution of Eq. (4) as a function of $k\rho_i$ for different values of L_s/L_n. The agreement is remarkable. The slight deviation for large L_s/L_n is due to the asymptotic nature of the analytic result. Both techniques predict stability ($\mathrm{Im}\omega < 0$) for L_s/L_n from 8 to 32

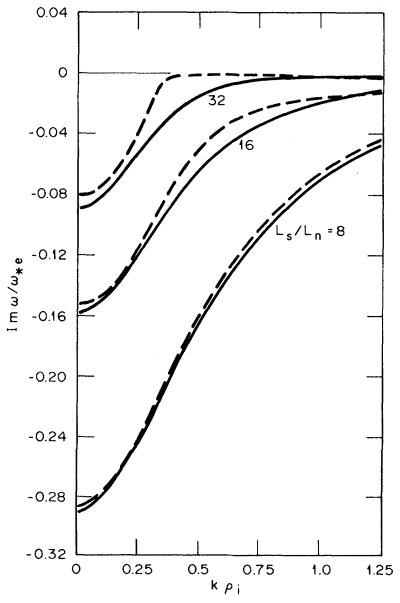

FIG. 1. Growth rate (normalized by ω_*) vs $k\rho_i$, for $T_e/T_i = 1$ and various values of L_s/L_n, obtained from shooting method with full electron Z function (solid line) and analytic dispersion relation (dashed line).

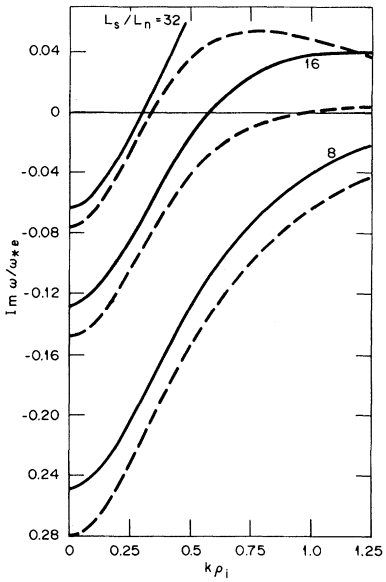

FIG. 2. Growth rate vs $k\rho_i$, for $T_e/T_i = 1$ and various values of L_s/L_n, obtained from shooting method with only the residue part of the electron Z function (solid line) and perturbation-theory dispersion relation (dashed line).

and $k\rho_i$ from 0 to 1.25. These results correct the prior numerical results presented in Fig. 1 of Ref. 3, which indicated instability. In fact, we have results for L_s/L_n up to 100 and $k\rho_i$ up to 10. All indicate that the mode is actually stable. The growth rate obtained from the perturbation-theory form of the eigenvalue equation, Eq. (3), is plotted vs $k\rho_i$ in Fig. 2 for various values of L_s/L_n. Instability is erroneously predicted for large $k\rho_i$ and L_s/L_n parameters. In the more strongly damped region, the perturbation theory agrees quite well with the numerical shooting result. This is consistent with the assumption of small σ_0 which is necessary to obtain Eq. (3) from Eq. (2). If the differential equation, Eq. (4), is numerically solved with only the residue of the electron Z function retained rather than the full Z function (as is proper), an even larger growth rate is found, also shown in Fig. 2. However, the eigenvalues are found to be insensitive to the large-phase-velocity approximation of the ion Z function; the eigenvalues of Eq. (1) are practically the same as those of Eq. (4).

Figure 3 shows a typical eigenfunction obtained from the numerical solution of Eq. (4) for the case $L_s/L_n=16$, $k\rho_i=1$, and $\tau=1$. Note that $(\mathrm{Re}\Phi)''$ is negative, which indicates that the distortion to the Weber solution caused by the resonant electrons is much weaker.

Recall that the residue of the electron Z function is the contribution of the resonant electrons. By comparing the shooting results of Figs. 1 and 2, we reach the important conclusion that the nonresonant electrons have a strong stabilizing influence and stabilize the instability in weak shear. The real part of ω is found to be reduced when the nonresonant part of the electron Z function is turned on.

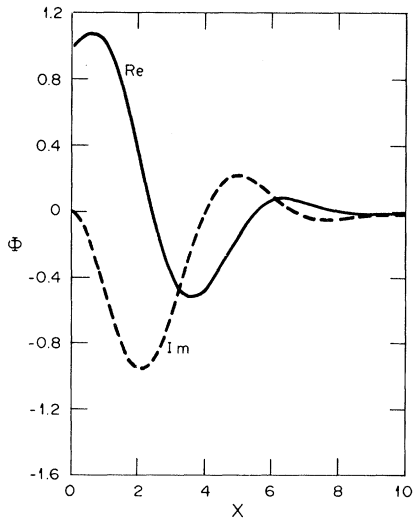

FIG. 3. $\mathrm{Re}\Phi$ and $\mathrm{Im}\Phi$ obtained from numerical solution of Eq. (4) with $L_s/L_n = 16$, $k\rho_i = 1$, and $T_e/T_i = 1$.

In Ref. 5, it is pointed out that Eq. (2) is obtained by an asymptotic expansion in which $|\sigma_0/\mu^{1/2}| \lesssim 1$ is assumed. If the next-order corrections to Eq. (2) are retained, the improved analytic eigenvalue equation gives marginally better agreement for the smaller L_s/L_n but far worse agreement for the larger L_s/L_n. This behavior is consistent with the asymptotic nature of the solution technique employed in Refs. 4 and 5. In addition, if the full Z function is replaced by its residue, the next order analytic corrections agree extremely well with the solid curves in Fig. 2 for small L_s/L_n.

In conclusion, we have found from an improved analytic eigenvalue equation and numerical solution of the appropriate differential equation that the collisionless drift wave in a sheared magnetic field is always stable for the parameter range of interest in tokamaks. This result is in contradiction to all previous work. The results of both of our techniques are in remarkable agreement. Stability is due primarily to the stabilizing influence of the nonresonant electrons (the principal-value part of the electron Z function). We must point out, however, the convective-universal or connective-collisionless drift instability has not been considered here and should be reconsidered in the light of these new results. Furthermore, the reader is cautioned that stability in the slab limit treated herein does not imply that stability will persist when trapped electrons and other toroidal effects such as ion drift are retained.

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Scaling of the Specific Heat of Confined Helium near T_λ

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We have measured the specific heat of helium near the superfluid transition for films up to 56 Å thick and for samples confined to a cylindrical geometry of up to 2000 Å in diameter. The dependence of the shift in the transition temperature on size and the growth of the specific-heat maximum do not yield exponents as expected from simple scaling arguments. The data up to $t \equiv |1 - T/T_\lambda| = 10^{-2}$ scale according to a form suggested by Fisher, but with an exponent $1/\theta = 0.54 \pm 0.02$.

Experimental studies of second-order phase transitions are typically done in the limit where the dimensions of the sample are much larger than the correlation length. For samples of about a millimeter in dimension, effects of sample geometry and boundary conditions at surfaces become important in a region too close to the transition to be experimentally detected. It is of considerable interest, on the other hand, to see how these effects become manifest, and to see if the presence of surfaces necessitates the introduction of new critical exponents.

Fisher[1] has pointed out that the specific heat, which for a bulk system displays a sharp transition at a temperature T_c, will become rounded to a maximum at a temperature T_m shifted from T_c. The extent of deviation from bulk behavior depends on the smallest confining dimension and defines a region of rounding. The maximum value at the specific heat will also depend on this confinement in a way which depends on the behavior of the specific heat in the bulk. These qualitative statements are summarized in the following equations:

$$d = a t_m^{-1/\Lambda}, \tag{1}$$

$$d = b(\delta t)^{-1/\theta}, \tag{2}$$

$$C_m = A\theta \ln d + \text{const}, \tag{3}$$

where d is the smallest confining dimension, $t_m = 1 - T_m/T_c$, δt is the region of rounding, and θ and Λ are some characteristic exponents. Equation (3) applies if the bulk has a specific heat which behaves as $C = A \ln|1 - T/T_c|$; i.e., the characteristic specific-heat exponent α is zero.

Since the length scale at the transition is set by the correlation length which diverges with a characteristic exponent ν, one would expect the exponents Λ and θ to be related to ν; specifically,

$$1/\theta = \nu, \tag{4}$$

$$1/\Lambda = \nu. \tag{5}$$

An alternative to Eq. (5) has also been conjectured[2,3]:

$$1/\Lambda > 1/\theta = \nu. \tag{6}$$

Liquid helium at the λ transitions is a good candidate to check on these predictions because of the ease with which one can achieve confinement in rather small dimensions. Indeed, experiments with helium films formed on various substrates or for helium confined in pores or packed powders have shown that the superfluid transition is shifted in temperature and the specific heat does lose its characteristic λ shape.[4-7] There have been, however, no quantitative verifications, via the specific heat, of the predictions for critical behavior. The reason for this has been mainly that one has not been able to control the confining geometry in a satisfactory way. In the case of films one runs into difficulties with capillary condensation leading to inhomogeneities in the thickness.[8,9] For confinement in packed powders, the size is too ill defined for a quantitative test. There have been a number of experiments to obtain a shift exponent Λ via the determination of the vanishing of the superfluid density.[10-13] These