ma device. The results of the computer simulation which we have presented indicate that the size of the "device" plays a role in determining which DL are stable.

We thank Alfred Scheller for construction and reconstruction of the triple plasma device. We also thank Dr. Stanley Shawhan and Dr. Nicola D'Angelo for a number of useful discussions and Dr. B. H. Quon for providing details of the experiment described in Ref. 3. This work was supported by the Atmospheric Research Section and by the Engineering Division of the National Science Foundation. One of us (R.H.) is a National Academy of Science/National Research Council Research Associate.

(a) Permanent address: Department of Physics and Astronomy, University of Iowa, Iowa City, Ia. 52242.

[1] F. S. Mozer, C. N. Carlson, M. K. Hudson, R. B. Torbert, B. Parady, J. Yatteau, and M. C. Kelley, Phys. Rev. Lett. 38, 292 (1977); E. M. Wescott, H. C. Stenback-Nielsen, T. J. Hallinern, and T. N. Davis, J. Geophys. Res. 81, 4495 (1976); L. P. Block, Cosmic Electrodyn. 3, 349 (1972); L. P. Block, private communication, 1977; S. D. Shawhan, C.-G. Fälthammer, and L. P. Block, University of Iowa Report No. 77-25, 1977 (unpublished).

[2] D. Anderson, Department of Electron Physics, Royal Institute of Technology, Sweden, Report No. TRITA-EPP-76-10, 1976 (unpublished); S. Torven and M. Babic, in *Proceedings of the Twelfth International Conference on Phenomena in Ionized Gases, Eindhoven, The Netherlands, 1975*, edited by J. G. A. Hölscher and D. C. Schram (American-Elsevier, New York, 1975), Pt. 1, p. 124.

[3] B. H. Quon and A. Y. Wong, Phys. Rev. Lett. 37, 1393 (1976).

[4] D. C. Montgomery and G. Joyce, J. Plasma Phys. 3, 1 (1969).

[5] D. W. Forslund and J. P. Friedberg, Phys. Rev. Lett. 27, 1189 (1971); J. S. DeGroot, C. Barnes, A. E. Walstead, and O. Buneman, Phys. Rev. Lett. 38, 1283 (1977).

[6] G. Knorr and C. K. Goertz, Astrophys. Space Sci. 31, 209 (1974).

[7] F. F. Chen, *Plasma Diagnostic Techniques*, edited by R. H. Huddlestone and S. L. Leonard (Academic, New York, 1965), p. 113.

[8] R. Hubbard and G. Joyce, to be published.

Korteweg–de Vries Soliton in a Slowly Varying Medium

K. Ko and H. H. Kuehl
University of Southern California, Los Angeles, California 90007
(Received 30 June 1977)

We derive an approximate solution to the Korteweg–de Vries equation with slowly varying coefficients for a soliton initial condition. Expressions are given for the amplitude, position, and velocity, and it is shown that the soliton experiences an irreversible loss of energy whenever it travels in a slowly varying medium. These results are applied to an ion acoustic soliton in a nonuniform plasma and are confirmed by comparison with the results of numerical integration of the differential equation.

The Korteweg–de Vries (KdV) equation arises in the study of various weakly nonlinear dispersive systems, e.g., shallow-water waves[1] and ion acoustic waves in plasmas.[2] Great interest in the KdV equation has been generated by the exact solutions found by Miura, Gardener, and Kruskal.[3] In many physical systems it is common that the medium in which a disturbance travels varies, e.g., the depth of a channel in which water waves travel is not constant, or the unperturbed plasma density in which an ion acoustic wave propagates is a function of position. We therefore present here the results of a theoretical and numerical study of a KdV soliton in a slowly varying medium. The theoretical technique is based on a perturbation expansion of the variable-coefficient KdV equation where we assume that the scale on which the soliton varies is short compared to that on which the medium varies. Other attempts[4,5] have been made to predict the behavior of a KdV soliton in a slowly varying medium but these are in error. Our solution differs from those of previous workers in that it shows for the first time that a KdV soliton loses energy whenever it travels in a slowly varying medium, independently of how the medium varies.

We consider a soliton governed by the KdV equation with slowly varying coefficients,

$$u_t + \alpha(T) u u_x + \beta(T) u_{xxx} = 0, \quad \alpha, \beta > 0, \tag{1}$$

where the coefficients α and β are arbitrary positive functions of a slow time variable $T = \epsilon t$, with

$\epsilon \ll 1$. The details of the perturbation theory that we use are given by Johnson,[4] Whitham,[6] Luke,[7] and Grimshaw.[8] Here we merely outline the essential steps. We seek an asymptotic solution of the form

$$u = u^{(0)}(\theta, T) + \epsilon u^{(1)}(\theta, T) + \epsilon^2 u^{(2)}(\theta, T) + \cdots, \quad (2)$$

where θ is the phase variable which reduces to $kx - \omega t$ when α and β are constants. Inserting Eq. (2) into Eq. (1) and collecting powers of ϵ, the terms of $O(\epsilon^n)$ give

$$[-\omega u^{(n)} + \alpha k(1+\delta_{0n})^{-1} u^{(0)} u^{(n)} + \beta k^3 u_{\theta\theta}^{(n)}]_\theta = F^{(n)}, \quad (3)$$

where δ_{0n} is the Kronecker δ and

$$F^{(0)} = 0, \quad F^{(1)} = -u_T^{(0)},$$

$$F^{(2)} = -u_T^{(1)} - \alpha k u^{(1)} u_\theta^{(1)}.$$

Upon integration of Eq. (3) with $n=0$, we obtain the KdV soliton

$$u^{(0)} = a\,\mathrm{sech}^2\varphi, \quad \varphi = b(\theta + \tilde{\theta}), \quad (4)$$

$$a = 3\omega/\alpha k, \quad b = (\omega/4\beta k^3)^{1/2}, \quad (5)$$

$$k(T) = \theta_x, \quad \omega(T) = -\theta_t, \quad (6)$$

and $\tilde{\theta}$ is an arbitrary function of T which was omitted by Johnson but is of crucial importance because its contribution causes the soliton to lose energy. From Eqs. (6), it follows that $k_t = -\omega_x = 0$ so that $k = k_0 = \text{const}$. Integrating Eq. (3) for $n=1$, we obtain

$$[\tfrac{3}{4}\int_{-\infty}^{\infty}(u^{(0)})^2 d\theta]_T = (a^2/b)_T = 0, \quad (7)$$

$$u^{(1)} = \frac{u^{(0)}\tilde{\theta}_T}{\omega}(\varphi\tanh\varphi - 1) - \frac{b_T u^{(0)}}{4\omega b^2}\{4(1+\varphi) - [4\varphi^2 + 6\varphi - \exp(-2\varphi) + D]\tanh\varphi\}, \quad (8)$$

where D is an arbitrary function of T. Inserting Eqs. (5) into Eq. (7), we obtain

$$\omega = \omega_0(\alpha/\alpha_0)^{4/3}(\beta/\beta_0)^{-1/3}, \quad (9)$$

where the zero subscripts denote initial values. Through the use of Eq. (9), it is noted that a, b, and θ are now completely determined. To determine $\tilde{\theta}$, we integrate Eq. (3) with $n=2$ to obtain

$$\tilde{\theta} = \tfrac{1}{2}(b^{-1} - b_0^{-1}) + \int_0^T \omega[\int_0^{T'}(b_{T''}{}^2/2\omega b^3)dT'']dT', \quad (10)$$

where we have assumed that at $t=0$, the medium does not vary and the initial disturbance is a pure soliton so that $b_T = \tilde{\theta} = \tilde{\theta}_T = 0$. The arbitrary function D in Eq. (8) can be obtained from Eq. (3) with $n=3$, but we do not present the result because D does not play a role in the results of interest here. From Eqs. (4) and (8), it may be verified that $|\epsilon u^{(1)}/u^{(0)}| \ll 1$ if $|\varphi| < 1$ and

$$\left|\frac{2b_t}{\omega b^2}\right| \ll 1, \quad \xi = \int_0^t \frac{b_t^2}{2\omega b^3}dt \ll 1. \quad (11)$$

The first inequality of Eqs. (11) states that the medium must vary on a time scale long compared to that on which the soliton varies. The second of Eqs. (11) requires that the fractional energy loss of the soliton be small [see Eq. (19)]. The present expansion is therefore useful near the soliton peak if Eqs. (11) hold.

Using Eqs. (4), (6), (8), and (10), we find to $O(\epsilon)$ the solution

$$u = u^{(0)} + \epsilon u^{(1)} = u_s + u_d, \quad (12)$$

where

$$u_s = a(1-\xi)\,\mathrm{sech}^2[(1-\xi)^{1/2}\varphi], \quad (13)$$

$$u_d = (-ab_t/2\omega b^2)\{1 + 2\varphi + \tfrac{1}{2}[\exp(-2\varphi) - 4\varphi - 4\varphi^2 - D]\tanh\varphi\}\,\mathrm{sech}^2\varphi, \quad (14)$$

$$\varphi = b[k_0 x - \int_0^t \omega(1-\xi)dt + \tfrac{1}{2}(b^{-1} - b_0^{-1})], \quad (15)$$

and ξ is defined in Eqs. (11). In Eq. (12), we have separated the disturbance into two parts: (a) u_s, which contains $u^{(0)}$ and the portion of $\epsilon u^{(1)}$ which does *not* vanish if the medium ceases to vary; u_s has the form (sech^2) of a soliton as well as the correct soliton relationship between the amplitude and width (which vary slowly when the medium varies); and (b) u_d, which is generated when the medium varies ($b_t \neq 0$) but vanishes after the medium ceases to vary ($b_t = 0$). We therefore designate u_s and u_d as the "soliton" and "soliton distortion," respectively. Since the maximum amplitude u_m of the disturbance occurs at $\varphi = 0$, we easily obtain from Eqs. (12)–(15)

$$u_m = a(1 - \xi - b_t/2\omega b^2), \tag{16}$$

$$x_m = k_0^{-1}[\int_0^t \omega(1-\xi)dt - \tfrac{1}{2}(b^{-1} - b_0^{-1})], \tag{17}$$

$$dx_m/dt = k_0^{-1}[\omega(1-\xi) + b_t/2b^2], \tag{18}$$

where x_m is the position of u_m. The most striking feature of our solution is that because of the quantity ξ, the soliton part of the disturbance loses energy whenever it travels in a slowly varying medium in spite of the fact that Eq. (1) contains no dissipative terms. This is clear if we calculate the soliton energy,

$$E_s = \int_{-\infty}^{\infty} u_s^2\, dx = (4a_0^2/3k_0 b_0)(1 - 3\xi/2), \tag{19}$$

where we have used Eq. (7). Because we have assumed $\alpha, \beta > 0$, it follows from the definition of ξ given by Eq. (11) that $\xi \geq 0$, and it is therefore evident from Eq. (19) that the soliton energy continually decreases when $b_t \neq 0$, independently of the sign of b_t. Moreover, this energy loss is *irreversible* in the sense that if α and β slowly change from their initial values α_0 and β_0 but then slowly revert to their initial values, the soliton does not revert to one with the initial amplitude, but rather one with an amplitude smaller by a factor of $1 - \xi$. From the results of numerical integration of Eq. (1) discussed subsequently, we find that the lost energy appears in the region behind the soliton. The present expansion is not useful to calculate the structure behind the soliton because the condition $|\varphi| < 1$ for our expansion to be valid is not satisfied.

In order to verify the predictions of this theory, we have numerically integrated Eq. (1) for the case of an ion acoustic soliton. Using a derivation similar to that of Nishikawa and Kaw[5] but with a different identification of variables, it can be shown that Eq. (1) governs one-dimensional nonlinear ion acoustic waves in an inhomogeneous plasma if $\alpha(t) = [N_0/N(t)]^{1/2}$, $\beta(t) = N_0/2N(t)$, $u(x,t) = [N(t)/N_0]^{1/2}[n_i(x,t)/N(t) - 1]$, where x and t represent $k_D x' - \omega_{pi} t'$ and $k_D x'$, respectively (x' and t' are the true distance and time variables, and k_D^{-1} and ω_{pi} are the Debye length and ion plasma frequency), and n_i and N are the ion density and unperturbed ion density, respectively. Using a finite-difference scheme, Eq. (1) was integrated for a density profile given by

$$N(t) = \tfrac{1}{2}\{N_1 + N_0 + (N_1 - N_0)\tanh[(t - t_0)/\tau]\}, \tag{20}$$

where t_0 is chosen such that $t_0 \gg \tau$ in order that $N = N_0$ at $t = 0$ (see Fig. 1). Computations were carried out for several values of final- to initial-density ratios N_1/N_0, density-profile time scales τ, and initial soliton amplitudes a_0. At $t = 0$, the soliton is given by $u_0 = a_0\,\text{sech}^2(b_0 k_0 x)$, where $b_0 k_0$ is determined from a_0 from Eqs. (5). Typical results for an increasing density profile are shown in Fig. 2. It is evident that the medium is undisturbed ahead of the soliton and that a shelflike formation develops behind the soliton during the time that the medium varies (an oscillatory tail is also present behind the shelf but its scale is too small to be shown on Fig. 2). However, for times sufficiently large such that the medium no longer varies, the shelf changes shape as it lags behind and eventually separates from the soliton. The shelf was also mentioned by Leibovich and Randall[9] and has been predicted in the theory of Kaup and Newell.[10] It is of interest to note that an appreciable time lag exists between the time at which the medium begins to vary appreciably and the time at which the shelf is formed.

A comparison of the results of perturbation theory with those from numerical integration is given in Fig. 3 which shows the maximum value of u at time t_0 (the time at which N varies most rapidly) and at large time ($t - t_0 \gg \tau$) vs the time scale τ on which the density profile varies. The curves obtained from numerical integration were constructed by taking the peak values of u at $t = t_0$ and at large t from curves such as those in Fig.

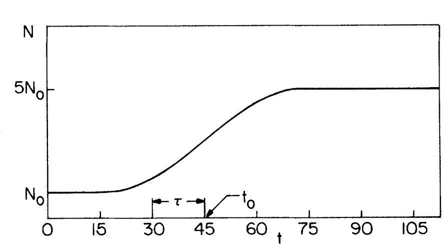

FIG. 1. Density profile given by Eq. (20) with $N_1 = 5N_0$, $t_0 = 45$, and $\tau = 15$.

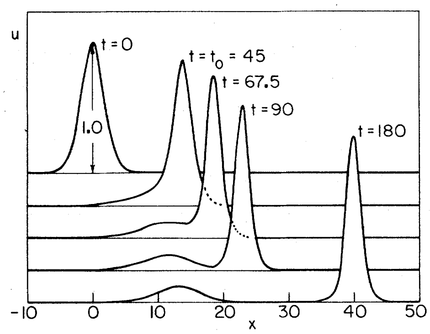

FIG. 2. Results of numerical integration of Eq. (1) for an ion acoustic soliton with initial amplitude $a_0 = 1$ for the density profile of Fig. 1. The curves are displaced vertically for clarity.

2 for various values of τ. The curves from perturbation theory were obtained from Eq. (16).

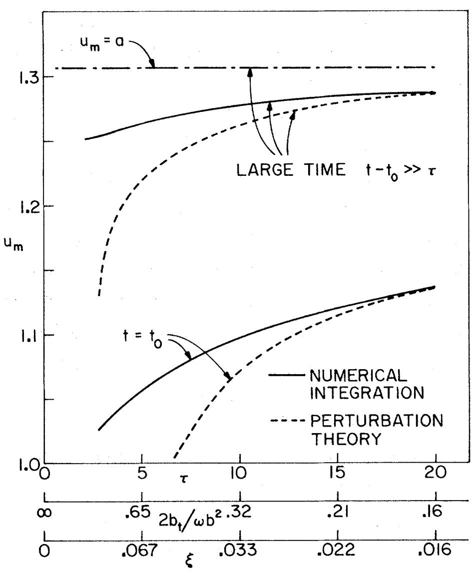

FIG. 3. Maximum value of u for an ion acoustic soliton with initial amplitude $a_0 = 1$ for the density profile of Eq. (20) with $N_1 = 5N_0$.

Also shown on the abscissa are the corresponding values of $2b_t/\omega b^2$ and ξ, from which it is clear that the perturbation result is an increasingly better approximation as Eqs. (11) are more closely satisfied, i.e., as the density varies more slowly. For the case of large time, Eq. (16) gives $u_m = a(1 - \xi)$, and for comparison we also show Johnson's result[4] $u_m = a$. Our results are consistently smaller than this because of the energy lost by the soliton.

We have also compared the position and velocity of the maximum of the disturbance given by Eqs. (17) and (18) with the results of numerical integration. Again good agreement is found when Eqs. (11) are satisfied. Finally, by calculating $\int_{-\infty}^{\infty} u^2\,dx$ for the case of large time where the trailing structure is separated from the soliton, e.g., $t = 180$ in Fig. 2, we have verified that the energy lost by the soliton is contained in this trailing structure.

This work was supported by the National Science Foundation.

[1]D. J. Korteweg and G. de Vries, Philos. Mag. 39, 422 (1895).

[2]H. Washimi and T. Taniuti, Phys. Rev. Lett. 17, 966 (1966).

[3]R. M. Miura, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. (N.Y.) 9, 1204 (1968).

[4]R. S. Johnson, J. Fluid Mech. 60, 813 (1973).

[5]K. Nishikawa and P. K. Kaw, Phys. Lett. 50A, 455 (1975).

[6]G. B. Whitham, Proc. Roy. Soc. London, Ser. A 283, 238 (1965).

[7]J. C. Luke, Proc. Roy. Soc. London, Ser. A 292, 403 (1966).

[8]R. Grimshaw, J. Fluid Mech. 42, 639 (1970).

[9]S. Leibovich and J. D. Randall, J. Fluid Mech. 53, 481 (1973).

[10]D. J. Kaup and A. C. Newell, to be published.