stant values of ν less than $1+t_0^2$, $dr/dt \leqslant (-1+\epsilon/2)$ [slope assumption in the transition region], and (iv) every curve beginning in the transition region with $t \geqslant t_0$ and having $\dot{r} \geqslant (-1+\epsilon/2)$ reaches I [convexity]. At any point along our geodesic with $t \geqslant t_0$ and $\nu \leqslant 1+t_0^2$, we must, by (i) and (ii), have $\dot{r} \geqslant (-1+\epsilon/2)$, and so, by (iii), $\dot{\nu} \leqslant 0$. Thus, $\nu \leqslant 1+t_0^2$, once achieved, is maintained; while by hypothesis it is achieved. We conclude that $\dot{r} \geqslant (-1+\epsilon/2)$ along some final segment of our geodesic, whence, by (iv), our geodesic reaches I.

We note, finally, that these cases exhaust the possibilities for null geodesics. Hence, all null geodesics reach I, completing the demonstration of weak asymptotic simplicity.

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[1]By a space-time we mean a smooth connected four-manifold with a smooth, time-oriented metric of Lorentz signature.

[2]R. Penrose, Proc. Roy. Soc. London, Ser. A 284, 159 (1965), and in *Battelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), pp. 121.

[3]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge Univ. Press, Cambridge, 1973).

[4]This condition essentially ensures that the generators of I are shear-free, and that the physical Ricci tensor vanishes sufficiently quickly asymptotically. The operations of taking derivatives, and of raising and lowering indices, are those with respect to the conformally scaled metric g_{ab}.

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[7]As usual, I^+ (respectively, I^-) denotes the set of points of I reached by future-directed (respectively, past-directed) timelike curves.

Brownian Motion of Coupled Nonlinear Oscillators: Thermalized Solitons and Nonlinear Response to External Forces

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We study the steady-state dynamical behavior of a set of torsion-coupled pendula in the presence of damping, fluctuating thermal torques, and constant applied torque. For small applied torque, the average angular velocity of the pendula at low temperature is associated with the motion of thermalized sine-Gordon solitons and as the torque is increased the velocity response becomes strongly nonlinear. These results can be used to describe nonlinear response in Josephson transmission lines and weakly pinned one-dimentional charge-density–wave condensates.

Recently there has been growing interest in condensed-matter systems characterized by nonlinear wave equations which possess solitary-wave or soliton solutions.[1-8] The equilibrium statistical mechanics of these systems has proven to be very interesting.[6,9,10] In this Letter we examine the problem of the nonequilibrium statistical mechanics of a ring of torsion-coupled pendula in a gravitational field which undergo Brownian motion in the presence of damping and are driven by external applied torque.[11] We use a transfer operator technique familiar from equilibrium statistical mechanics[6,9,12] to solve the Smoluchowski equation and find the average steady-state angular velocity, $\bar{\omega}$, of the pendula as a function of temperature and applied torque. This average angular velocity is a physically relevant quantity in several contexts; e.g., it is the mean thermal noise voltage[13] across a Josephson transmission line,[5] the average dc current density in a weakly pinned one-dimensional charge-density-wave (CDW) condensate,[7] etc. We find that for small applied torques, $\bar{\omega}$ is proportional to the number of thermally activated solitons present in the system. As the applied torque is increased toward the value necessary to overcome the restoring force of the gravitational field, $\bar{\omega}$ rises sharply and is a nonlinear function of the torque. This behavior is qualitatively similar to that found[14] for the current density as a function of applied electric field in materials believed to possess pinned CDW's.

Consider a system of N ($N \gg 1$) simple pendula of mass m and length l whose points of support are equally spaced on a large supporting ring. Each pendulum is coupled to its nearest neighbors by a torsion spring (torsion constant κ) and is free to move only in the vertical plane containing its point of support and the center of the support ring. The motion of each pendulum can thus be described by an angular coordinate θ (measured from the vertical) and an angular velocity $\dot{\theta} \equiv d\theta/dt$. The classical Lagrangian for this system in a gravitational field can be written as ($L = T - U$):

$$L = \sum_{i=1}^{N} \{ \tfrac{1}{2} ml^2 \dot{\theta}_i^2 - mgl(1 - \cos\theta_i) - \tfrac{1}{2}\kappa(\theta_{i+1} - \theta_i)^2 + \tau\theta_i \}, \tag{1}$$

where g is the acceleration due to gravity, and τ is a constant torque applied to each pendulum. The ring configuration for this system means $\theta_{N+1} = \theta_1$.[15]

In terms of angular momenta, $p_i \equiv ml^2\dot{\theta}_i$, we write the Langevin equation of motion for the ith pendulum:

$$\dot{p}_i = K(\theta_{i-1}, \theta_i, \theta_{i+1}) - \eta p_i + F_i(t), \tag{2}$$

where

$$K(\theta_{i-1}, \theta_i, \theta_{i+1}) = \kappa(\theta_{i+1} + \theta_{i-1} - 2\theta_i) - mgl \sin\theta_i + \tau = -\partial U(\theta_1 \ldots, \theta_N)/\partial\theta_i.$$

The last two terms in Eq. (2) represent phenomenological damping and fluctuating "noise" torque, respectively. The noise is assumed to be thermal and to act on each pendulum independently so that $\langle F_i(t)\rangle = 0$ and $\langle F_i(t)F_j(t+t')\rangle = 2ml^2 k_B T\eta\,\delta_{ij}\delta(t')$.

In the steady-state situation, the average angular velocity of each pendulum will be the same, i.e., $\langle\dot{\theta}_i\rangle \equiv \bar{\omega}$ for all i. To calculate $\bar{\omega}$, we start from a multidimensional Fokker-Planck equation[16] for the phase-space distribution function $P(\theta_1, \ldots, \theta_N; p_1, \ldots, p_N; t)$:

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{N} \left\{ -K(\theta_{i-1}, \theta_i, \theta_{i+1}) \frac{\partial P}{\partial p_i} - \frac{p_i}{ml^2}\frac{\partial P}{\partial \theta_i} + \eta \frac{\partial}{\partial p_i}\left(p_i P + ml^2 k_B T \frac{\partial P}{\partial p_i}\right)\right\}. \tag{3}$$

Because of the nonlinear part ($\sin\theta_i$) of the torque K, Eq. (3) is extremely difficult to solve in general. However, in the limit when the damping constant, η, is large[17] compared to the characteristic frequency of the pendula, $\omega_0 = \sqrt{g/l}$, we can use the method of Kramers[18] to integrate over the momenta in Eq. (3), yielding a multidimensional Smoluchowski equation for the coordinate distribution function $\sigma(\theta_1, \ldots, \theta_N; t)$:

$$\frac{\partial \sigma}{\partial t} = \frac{2\omega_0^2}{\gamma\eta} \sum_{i=1}^{N} \frac{\partial}{\partial\theta_i}\left\{ e^{-\beta U} \frac{\partial}{\partial \theta_i}[e^{\beta U}\sigma]\right\}, \tag{4}$$

where $U(\theta_1, \ldots, \theta_N)$ is the total potential [see Eq. (1)]. We introduce the following definitions for convenience: $\beta \equiv (k_B T)^{-1}$, $\gamma \equiv 2\beta mgl$, $d \equiv (\kappa/mgl)^{1/2}$, and $\chi \equiv \tau/\tau_c$ with $\tau_c \equiv mgl$. The dimensionless parameter γ is the ratio of the gravitational potential barrier height to thermal energy, d is a characteristic length scale (the "width" of the soliton excitation[11] measured in numbers of pendula), and τ_c is the critical torque required to give a nonzero average angular velocity at $T = 0$.

To find the average steady-state angular velocity, $\bar{\omega}=\langle\dot{\theta}_i\rangle$, we single out one of the angles (say θ_j) and integrate Eq. (4) over all other angles to obtain an equation for the single-particle distribution function $\sigma(\theta_j)=\int d\theta_1\ldots\int d\theta_{j-1}\int d\theta_{j+1}\ldots\int d\theta_N\,\sigma(\theta_1,\ldots,\theta_N;t)$. In the steady state we have $\partial\sigma(\theta_1,\ldots,\theta_N)/\partial t=0$ and, in particular,

$$\partial\sigma(\theta_j)/\partial t=0\equiv-\partial w/\partial\theta_j, \tag{5}$$

where w is a constant diffusion current. Since $\sigma(\theta_j)$ is periodic $[\sigma(\theta_j+2\pi)=\sigma(\theta_j)]$, we consider the interval $0\leq\theta_j\leq2\pi$ and normalize $\sigma(\theta_j)$ by the condition $\int_0^{2\pi}d\theta_j\,\sigma(\theta_j)=1$. With this condition, w^{-1} is the average time required for θ_j to evolve by 2π, hence $\bar{\omega}=\langle\dot{\theta}_j\rangle=2\pi w$.

In order to solve Eq. (4) we write $\sigma(\theta_1,\ldots,\theta_N)$ in factored form as

$$\sigma(\theta_1,\ldots,\theta_N)=\rho(\theta_1,\ldots,\theta_N)h(\theta_1,\ldots,\theta_N),$$

where $\rho(\theta_1,\ldots,\theta_N)=\exp[-\beta U(\theta_1,\ldots,\theta_N)\,|_{\tau=0}]$ is the zero-torque ($\tau=0$) equilibrium distribution function and $h(\theta_1,\ldots,\theta_N)$ contains the effects of the external torque and remains to be determined. We make the *Ansatz*[19] $h(\theta_1,\ldots,\theta_N)=h(\theta_1)h(\theta_2)\ldots h(\theta_N)$, where $h(\theta_i)$ is a periodic single-particle function. This allows us to integrate Eq. (4) over all angles except θ_j with the aid of a transfer operator technique. Using Eq. (5) we find

$$w=(\omega_0^2/\eta)\sigma(\theta_j)[\chi-\partial y(\theta_j)/\partial\theta_j], \tag{6}$$

where $y(\theta)\equiv(2/\gamma)\ln h(\theta)$. The single-particle distribution function can be expressed as

$$\sigma(\theta)=\textstyle\sum_k e^{-\gamma N\epsilon_k/2}|\Phi_k(\theta)|^2, \tag{7}$$

where ϵ_k and Φ_k are the eigenvalues and associated engenfunctions of the transfer integral operator;

$$\int_{-\infty}^{+\infty}d\theta_j\exp[-\tfrac{1}{2}\gamma v(\theta_{j+1},\theta_j)]\Phi_k(\theta_j)=\exp[-\tfrac{1}{2}\gamma\epsilon_k]\Phi_k(\theta_{j+1}), \tag{8}$$

with

$$v(\theta_{j+1},\theta_j)\equiv\tfrac{1}{2}[d^2(\theta_{j+1}-\theta_j)^2-\cos\theta_j-\cos\theta_{j+1}-y(\theta_j)-y(\theta_{j+1})].$$

In the thermodynamic limit ($N\to\infty$), only the ground state is important in Eq. (7) and thus $\sigma(\theta)=|\Phi_0(\theta)|^2$ (normalizing so that $\epsilon_0=0$). This result, together with Eq. (6), yields the dimensionless average angular velocity $\Omega\equiv\bar{\omega}\eta/\omega_0^2$:

$$\Omega=4\pi^2\chi[\int_0^{2\pi}\sigma^{-1}(\theta)d\theta]^{-1}. \tag{9}$$

The ground-state eigenfunction contained in $\sigma(\theta)$ $[=|\Phi_0(\theta)|^2]$ is determined self-consistently by Eq. (8) and the first integral of Eq. (6):

$$y(\theta)=\chi\left\{\theta-2\pi\left[\int_0^{2\pi}\frac{d\theta'}{\sigma(\theta')}\right]^{-1}\int_0^{\theta}\frac{d\theta''}{\sigma(\theta'')}\right\}. \tag{10}$$

In the strong-coupling limit, d becomes large and the Fredholm integral equation (8) for the ground-state eigenfunction $\Phi_0(\theta)$ can be approximated[12,6,9] by a differential equation for a related function $\psi_0(\theta)\equiv\exp\{\tfrac{1}{4}\gamma[\cos\theta+y(\theta)]\}\Phi_0(\theta)$:

$$[-(1/2\mu)(d^2/d\theta^2)-\cos\theta-y(\theta)]\psi(\theta)=\epsilon\psi(\theta), \tag{11}$$

where $\mu\equiv(\tfrac{1}{2}\gamma d)^2$. Equation (11) has the form of Schrödinger's equation ($\hbar=1$) for a particle of "mass" μ in a periodic potential. The solutions have the Floquet form, $\psi_k(\theta)=\exp(ik\theta)u_k(\theta)$ with $u_k(\theta+2\pi)=u_k(\theta)$, and the eigenvalues form bands in k space. We need only the lowest state, corresponding to the bottom ($k=0$) of the lowest band.

In Fig. 1 we plot the average angular velocity Ω as a function of the torque χ, for several values of the temperature (measured in units of $2mgl$, $k_BT=\gamma^{-1}$), from the self-consistent solution of Eqs. (10) and (11). At low torque Ω is linear in the torque and proportional to the number of thermally activated solitons (see below); at high torque the current is again linear as the pendula are driven by the torque independent of the strength of the gravitational field, mgl, and the coupling, κ. As the temperature is raised Ω increases rapidly as the number of solitons increases. At fixed γ^{-1}, increases in χ lead to nonlinear evolution of Ω.

In the limit $\chi\ll1$, the χ dependence of $\Phi_0(\theta)$ may be neglected and Ω becomes linear in χ [Eq. (9)]. The function $y(\theta)$ may be neglected and Eq. (11) becomes the Mathieu equation.[20] From the asymptotic properties of the ground-state Mathieu

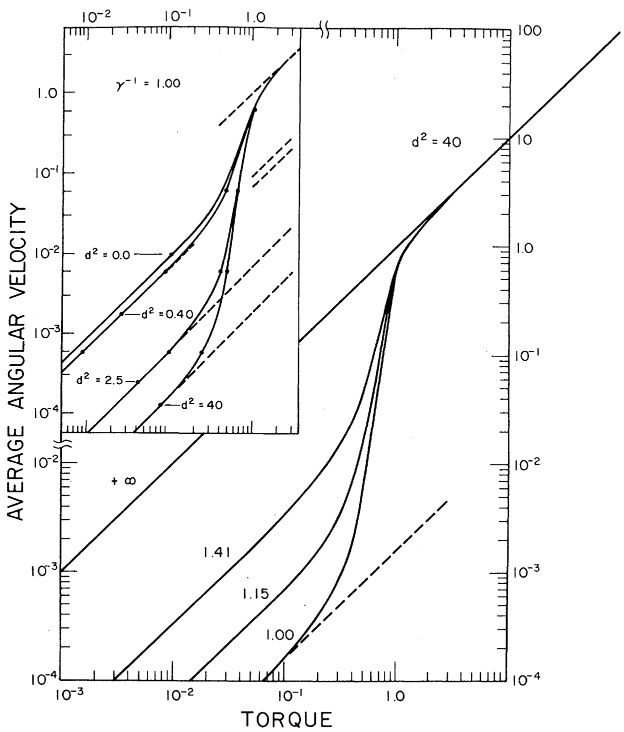

FIG. 1. The average angular velocity vs torque. The average angular velocity is calculated from the self-consistent solution of Eqs. (10) and (11) for three choices of the temperature (measured by γ^{-1}). Inset: Ω vs χ at fixed temperature ($\gamma^{-1}=1$) as the strength of the interpendulum coupling (measured by d^2) is increased from 0 to $d^2=40$. The strong-coupling approximation [Eq. (11)] is in error by about 20% for $d^2=0.4$ but very accurate for $d^2=2.5$ and 40.

function, we find that at low temperatures

$$\frac{\Omega}{\chi} \simeq 2\pi\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{E_0}{k_B T}\right)^{3/2}\exp\left[-\left(1-\frac{1}{8d}\right)\frac{E_0}{k_B T}\right]$$

$$(E_0 \gg k_B T;\ \chi \ll 1), \qquad (12)$$

where $E_0 = 8mgld$ is the rest energy of the sine-Gordon soliton.[1] The sine-Gordon soliton is a solution to the equation of motion for the undriven system in the strong-coupling limit ($d \gg 1$). The equilibrium (zero-torque) density of solitons (plus antisolitons) is given by[10]

$$n(T) = 2(2/\pi)^{1/2} d^{-1}(E_0/k_B T)^{1/2}\exp(-E_0/k_B T).$$

Thus we see that in the limit $\chi \ll 1$ and $E_0/k_B T \gg 1$, Ω is proportional to $n(T)$, the density of solitons.[7]

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function σ to the equilibrium ($\tau = 0$) distribution function ρ. The N-particle Smoluchowski equation can be treated in terms of a sequence of equations analogous to the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy.

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Solution of Multiple Scattering by Finite Iteration

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We propose an iterative nonlinear solution to the general potential scattering problem in quantum mechanics. It is illustrated by the case of N_I localized scatterers of arbitrary strengths V_j located at arbitrary points R_j in an infinite lattice, for which we obtain the complete set of bound and scattering states. The numberical evaluation is estimated to take $O(N_I)$ steps as opposed to $O(N_I^3)$ steps by conventional matrix inversion.

Scattering theory, as traditionally expressed in the derivation and solution of the equation[1]

$$\psi(\vec{r}) = \eta_E \psi^{(0)}(\vec{r}) - \int G_E^{(0)}(\vec{r}\,', \vec{r}) V(\vec{r}\,')\psi(\vec{r}\,') d^3r' \tag{1}$$

with $\eta_E = 1$ for E in the continuum and 0 for bound states, is generally intractable unless the potential is weak or has some high symmetry. For an arbitrary potential where the usual[1] expansions are inadequate and for which no simplifying symmetry is ascertained, I propose the following nonlinear, nonperturbative approach. The idea is to solve for the contribution from a small neighborhood $\Delta\Omega$ of each individual point where $V(\vec{r}\,')$ differs from zero, one point at a time. For example, starting at a specific $\vec{r}_0$ one would first consider

$$\psi(\vec{r}) = \psi^{(0)}(\vec{r}) - G_E^{(0)}(\vec{r}_0, \vec{r}) V(\vec{r}_0)\psi(\vec{r}_0)\Delta\Omega,$$

which has the explicit solution

$$\psi^{(\vec{r}_0)}(\vec{r}) \equiv \psi(\vec{r}) = \psi^{(0)}(\vec{r}) - \frac{\Delta\Omega V(\vec{r}_0) G_E^{(0)}(\vec{r}_0, \vec{r})\psi^{(0)}(\vec{r}_0)}{1 + G_E^{(0)}(\vec{r}_0, \vec{r}_0) V(\vec{r}_0)\Delta\Omega}, \tag{2}$$

and use this as the input at the next point $\vec{r}_1$

$$\psi^{(\vec{r}_0, \vec{r}_1)}(\vec{r}) = \psi^{(\vec{r}_0)}(\vec{r}) - \frac{\Delta\Omega V(\vec{r}_1) G_E^{(1)}(\vec{r}_1, \vec{r})\psi^{(\vec{r}_0)}(\vec{r}_1)}{1 + G_E^{(1)}(\vec{r}_1, \vec{r}_1) V(\vec{r}_1)\Delta\Omega}, \tag{3}$$

in which there appears a new Green's function $G^{(1)}$ constructed with the eigenfunctions (2) and thus incorporating $V(\vec{r}_0)$. This procedure is then iterated, but it presents some difficulties. In addition to the scattering states there may appear bound states which must be computed separately. If we proceed to the limit $\Delta\Omega \to 0$, the number of points at which we must iterate becomes infinite. A proper formulation undoubtedly involves differential quantities such as $\partial\psi/\partial V$ at each point. Finally, the ultraviolet divergence of $G(\vec{r}, \vec{r})$ in two, three, or more dimensions necessitates a high-energy cutoff, which may be allowed to go to infinity only at the end of the calculations. At the present time I do not know how to circumvent these difficulties, which appear to provide many opportunities for further investigation. Nevertheless the solid-state analog to this problem is completely and explicitly solvable by such an iterative technique, as I now show.

We consider a simplified case, where electrons are confined to a single energy band of a solid with $N = \infty$ atoms, with N_I arbitrarily placed localized scatterers diffracting the electron waves. We shall obtain the exact eigenstates by a succession of N_I rotations of the Hilbert space. The model incorporates two important simplifications: First, the finite bandwidth of Bloch energies ϵ_k confined to a single band ensures that the Green's functions are free of ultraviolet divergences, obviating the need for an artificial cutoff. Second, the discrete nature of point scatterers enables us to terminate the process after a denumerable N_I steps.

We recall the few facts and the notation which are almost all the reader will have to know of solid

