PHYSICAL REVIEW LETTERS

VOLUME 40 | 23 JANUARY 1978 | NUMBER 4

Asymptotically Simple Does Not Imply Asymptotically Minkowskian

Robert Geroch and Gary T. Horowtiz
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
(Received 31 October 1977)

An example is given of a space-time which satisfies the conditions for asymptotic region is not "as large" as that of Minkowski space-time.

There is available in general relativity an elegant and useful treatment of isolated systems, i.e., of space-times which possess an asymptotic regime similar to that of Minkowski space-time. This treatment is based on the following two definitions. A space-time[1] $\tilde{M}$, $\tilde{g}_{ab}$ is said to be *asymptotically simple*[2,3] if there exists a manifold with boundary, $M=\tilde{M}\cup I$, consisting of $\tilde{M}$ with boundary I (the ideal points "at null infinity") attached, together with smooth Lorentz metric g_{ab} and smooth scalar field Ω on M, such that (1) on $\tilde{M}$, $g_{ab}=\Omega^2\tilde{g}_{ab}$; (2) at I, Ω vanishes, its gradient is nonzero and null, and its second derivative[4] vanishes; (3) every maximally extended null geodesic in $\tilde{M}$ has, in M, two endpoints on I. *Weak asymptotic simplicity*[2,3] is defined similarly, but with the third condition replaced by another: (3′) There exists a neighborhood of I in M which is g_{ab}-isometric to a neighborhood of the boundary of some asymptotically simple space-time.

The role of the first two conditions above is to impose the correct local asymptotic behavior on the gravitational field, i.e., the correct behavior as one recedes from the isolated source in a given null direction. The role of conditions (3) and (3′), by contrast, is to fix the global asymptotic behavior. They are intended to ensure that the asymptotic structure be globally the same as that of Minkowski space-time; that in particular the boundary I be as "large" as the boundary appropriate to Minkowski space-time. The distinction between 3 and 3′ is the following. Condition 3 is the stronger: It in addition imposes severe nonasymptotic global conditions on the space-time. This is the key condition, for example, in proving that an asymptotically simple space-time must possess a Cauchy surface, that it must be topologically R^4, etc. Condition (3′), on the other hand, retains only the purely asymptotic aspects of condition (3), for it requires essentially that in the asymptotic regime, i.e., in some neighborhood of I, the space-time be indistinguishable from an asymptotically simple one. Thus, for example, Minkowski space-time with the point at the origin removed is weakly asymptotically simple but not asymptotically simple.

That conditions (3) and (3′) do, in fact, capture the idea that the space-time must have the global asymptotic structure of Minkowski space-time seems to have acquired the status of a "folk theorem." That is, although such a result has, apparently, never been claimed explicitly in the literature, it has implicitly entered various definitions, conjectures, and discussions utilizing asymptotic structure. We shall here show that, by what seems to be a reasonable reading, this "folk theorem" is simply false. More precisely, we shall prove this:

Theorem. Consider Minkowski space-time with the causal future of the origin removed [i.e., retain the region given, in the usual coordinates, by $t<(x^2+y^2+z^2)^{1/2}$]. This space-time is weakly asymptotically simple.

It follows in particular from the theorem that, while of course Minkowski space-time with its usual boundary satisfies the conditions for weak asymptotic simplicity, so does Minkowski space-time with only a portion of that boundary (namely, that portion which lies outside the causal future of a point of the space-time).

A few examples will illustrate the relevance of this theorem to various applications of asymptotic structure. A space-time is said to possess a black hole[3] if it is weakly asymptotically simple and the past of future null infinity I^+ is not the entire space-time. This definition now acquires an ambiguity: Does one wish to demand that the past of I^+ be not the entire space-time for *some* I^+ which makes the space-time weakly asymptotically simple, or for *all* such I^+? If the resolution is "for some I^+," then Minkowski space-time would possess a black hole. If "for all I^+," then Minkowski space-time with an asymptotic portion of the null cone of the origin removed possesses a black hole (for every I^+ which makes this space-time weakly asymptotically simple must, of course, be connected). A space-time is said to be future asymptotically predictable[3] if it is weakly asymptotically simple and possesses an achronal slice S whose future domain of dependence in M includes I^+. Again, the nonuniqueness of I^+ leads to an ambiguity for which there seems to be no appropriate resolution. The Bondi-Metzner-Sachs group, if defined[5] as the group of diffeomorphisms on future null infinity I^+ of a weakly asymptotically simple space-time which preserve the intrinsic geometrical structure of I^+, would now, depending on the I^+, be several different groups, some of which have no supertranslations. One version of cosmic censorship conjectures[6] that the maximal vacuum evolution of any asymptotically flat initial-data set is weakly asymptotically simple. By a slight modification of the argument of the theorem, one sees that this conjecture is now true—but it now does not seem to say cosmic censorship.

It is easy to modify conditions (3) and (3′) so that they do indeed demand the global asymptotic structure of Minkowski space-time. Set $n^a = \nabla^a \Omega$, so that, by condition (2), n^a is the field of null generators of I. It is further consequence of condition (2) that the completeness or incompleteness of the vector field n^a on I is gauge invariant, i.e., is preserved by any new choice of the conformal factor Ω subject to this condition. The global asymptotic structure of Minkowski space-time is characterized by the structure of its usual boundary I, which in turn is characterized by its "shape," $S^2 \times \mathrm{R}$, and its "size," completeness of n^a on I. This suggests a definition: A space-time $\tilde{M}$, $\tilde{g}_{ab}$ will be said to be *asymptotically flat* (at null infinity) if there exists a manifold with boundary, $M = \tilde{M} \cup I$, consisting of $\tilde{M}$ with boundary I attached, together with smooth Lorentz metric g_{ab} and smooth scalar field Ω on M, such that (1) on $\tilde{M}$, $g_{ab} = \Omega^2 \tilde{g}_{ab}$; (2) at I, Ω vanishes, its gradient is nonzero and null, and its second derivative[4] vanishes; (3″) I consists of two parts,[7] I^+ and I^-, each $S^2 \times \mathrm{R}$ with the "R's" the null generators, and on each of which n^a is complete. That is to say, the earlier condition (3) or (3′) is replaced by (3″). The definition refers only to the asymptotic, and not to the internal, structure of the space-time. Thus, asymptotic flatness, which implies weak asymptotic simplicity, serves as its replacement, e.g., in the above examples. Instead of asymptotic simplicity, one can now just impose where appropriate, in addition to asymptotic flatness, further internal global conditions on the space-time, e.g., that it be R^4, that it admit a Cauchy surface, that condition (3) above be satisfied. As examples, Minkowski space-time and the Schwarzschild space-time are asymptotically flat, while the space-time of the theorem is not, for completeness there fails. More generally, by considering only I^+ or only I^- in (3″), one could define *future* or *past* asymptotic flatness.

Finally, we outline the proof of the theorem. Consider Minkowski space-time with its flat metric η_{ab} and a constant, unit, timelike vector field t^a. Consider the region of this space-time given [in the usual coordinates, with t_a the gradient of t and $r = (x^2 + y^2 + z^2)^{1/2}$] by $t \leq -r + 1$ and $t < 0$, i.e., the past light cone of the point labeled (+1, 0, 0, 0), with its "top chopped off" (Fig. 1). The

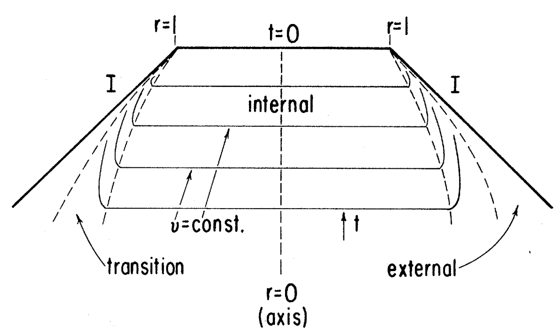

FIG. 1. A space-time with boundary I, in the r-t plane. The internal, transition, and external regions, as well as the curves of constant ν, are shown.

null boundary I is given by $t=-r+1<0$. Let the metric on this manifold with boundary be $g_{ab}=\eta_{ab}+(1-\nu^2)t_a t_b$, where ν is a smooth positive scalar field on M. Thus, large ν corresponds to light cones "opened out" about t^a; small ν to "closed in." We wish to choose this field ν such that $\nu=1$ (i.e., $g_{ab}=\eta_{ab}$) in a neighborhood of I, and such that every future-directed null geodesic in the space-time, not a generator of I, has future endpoint on I. The demonstration of the existence of such a ν, we claim, will complete the proof of the theorem. To see this, first recall that, for the usual conformal completion of Minkowski space-time, its top part with its conformally scaled metric can be isometrically embedded in Minkowski space-time, with I^+ the past null cone of a point. Hence, the space-time above will represent the top part of the space-time of the theorem, with its boundary at future null infinity attached and with the metric so modified outside a neighborhood of I that condition (3), i.e., that asymptotic simplicity, is satisfied.

The idea of the proof is simply to choose ν to increase as t approaches zero in order to open out the light cones, and thus force the null geodesics away from the central region toward I.

Divide the space-time into three regions, as shown in the figure. In the external region, which includes a neighborhood of I, set $\nu=1$. In the internal region, set $\nu=\nu_0=t^{-2}$. Between the internal and external is the transition region, in which $r>1$, and whose boundaries are to be convex inward and to approach I tangentially as $t\to 0$. Choose ν in this region to be spherically symmetric, such that $\nu\geq 1$, $\partial\nu/\partial r\leq 0$, such that the r-t slope of the curves of constant ν approaches -1 as $t\to 0$, and such that, at every point of this region, either $\nu\leq 1+t^2$ or $\nu-r\,\partial\nu/\partial r\geq\nu_0$. (This latter condition is accomplished by choosing ν, for constant t and increasing r, to decrease sufficiently rapidly that the second inequality is satisfied, until ν has decreased below $1+t^2$, at which point the first inequality is satisfied, and so ν can decrease more slowly to its final value of 1.)

We shall show that, with these conditions on ν, every future-directed null geodesic in our space-time reaches I. Parametrize such a geodesic by the (not necessarily affine) parameter t, and denote by a dot, applied to a function defined along the curve, its rate of change with respect to this parameter. Then, for example, nullness implies $\dot r^2\leq\nu^2$. Set $L=r^2(1-\nu^{-2}\dot r^2)$, a non-negative "effective squared angular momentum" along the geodesic. The null geodisic equation in the metric g_{ab} then yields

$$(\nu^{-1}\dot r)^{\cdot}=r^{-3}(\nu-r\,\partial\nu/\partial r)L, \qquad (1)$$

from which there follows immediately

$$\dot L=2\nu^{-1}\dot r(\partial\nu/\partial r)L. \qquad (2)$$

We consider various classes of null geodesics as follows.

(1) If the geodesic remains entirely in the external region, or emerges from the transition into the external, then it reaches I. This follows, since the space-time is flat in the external region, from convexity of the boundary of the transition region.

(2) If $\dot r$ is positive at any point along the geodesic, then that geodesic reaches I. Since the right-hand side of (1) is non-negative, positivity of $\dot r$ at any point of the geodesic implies that thereafter $\dot r\geq c\nu$, for some positive constant c. Therefore, the geodesic could not thereafter remain in the internal region, for there $\nu=\nu_0=t^{-2}$, whence the time integral of $c\nu$ to $t=0$, must reach the transition region, where $r>1$, and so must have $r>1$ thereafter, and so must reach I.

(3) If $L=0$ [i.e., the geodesic is radial, a condition which, by (2), is maintained along the geodesic], then that geodesic reaches I. Set, without loss of generality, $\dot r\leq 0$, so that $L=0$ implies $\dot r=-\nu$. But now, since $\dot r\leq -1$, the geodesic, if it leaves the external region at all, must enter the internal. But in the internal region, the time-integral of $\dot r=-\nu$ diverges, whence the geodesic must pass through the origin, and so attain positive $\dot r$, and so, by item (2), reach I.

(4) If the geodesic satisfies $\nu>1+t^2$ along some final segment, then it reaches I. Let, without loss of generality, $\dot r\leq 0$ and $L\neq 0$, so that, since the right-hand side of (2) is non-negative, $L\geq L_0$, for some positive constant L_0, along this final segment. But now, by hypothesis and the choice of ν in the transition region, the right-hand side of (1) exceeds $r^{-3}\,\nu_0 L_0$, whence the time integral of this right-hand side diverges. But this in turn means that $\dot r$ must somewhere become positive, and so, by item 2, the geodesic must reach I.

(5) If the geodesic fails to satisfy $\nu>1+t^2$ everywhere along some final segment, then that geodesic reaches I. Setting, without loss of generality, $L\neq 0$, we have by (1), for some positive ϵ, $\dot r\geq(-1+\epsilon)\nu$ along some final segment. Choose t_0 sufficiently near zero that (i) $t\geq t_0$ is within this final segment, (ii) $(-1+\epsilon)(1+t_0^2)\geq(-1+\epsilon/2)$, (iii) for $t\geq t_0$ we have, along the curves of con-

stant values of ν less than $1+t_0^2$, $dr/dt \leqslant (-1+\epsilon/2)$ [slope assumption in the transition region], and (iv) every curve beginning in the transition region with $t \geqslant t_0$ and having $\dot{r} \geqslant (-1+\epsilon/2)$ reaches I [convexity]. At any point along our geodesic with $t \geqslant t_0$ and $\nu \leqslant 1+t_0^2$, we must, by (i) and (ii), have $\dot{r} \geqslant (-1+\epsilon/2)$, and so, by (iii), $\dot{\nu} \leqslant 0$. Thus, $\nu \leqslant 1+t_0^2$, once achieved, is maintained; while by hypothesis it is achieved. We conclude that $\dot{r} \geqslant (-1+\epsilon/2)$ along some final segment of our geodesic, whence, by (iv), our geodesic reaches I.

We note, finally, that these cases exhaust the possibilities for null geodesics. Hence, all null geodesics reach I, completing the demonstration of weak asymptotic simplicity.

We wish to thank Bob Wald for helpful discussions. This work was supported in part by the National Science Foundation under Contract No. PHY-76-81102. One of us (G.T.H.) acknowledges receipt of a National Science Foundation predoctoral fellowship.

[1]By a space-time we mean a smooth connected four-manifold with a smooth, time-oriented metric of Lorentz signature.

[2]R. Penrose, Proc. Roy. Soc. London, Ser. A 284, 159 (1965), and in *Battelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), pp. 121.

[3]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge Univ. Press, Cambridge, 1973).

[4]This condition essentially ensures that the generators of I are shear-free, and that the physical Ricci tensor vanishes sufficiently quickly asymptotically. The operations of taking derivatives, and of raising and lowering indices, are those with respect to the conformally scaled metric g_{ab}.

[5]R. Penrose, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. DeWitt (Blackie & Sons, London, 1964), pp. 583; B. Schmidt, M. Walker, and P. Sommers, Gen. Rel. Grav. 6, 489 (1975).

[6]S. W. Hawking, in Proceedings of the Eighth International Conference on General Relativity and Gravitation, Waterloo, Canada, 7–13 August 1977 (to be published); G. T. Horowtiz, senior thesis, Princeton University, 1976 (unpublished).

[7]As usual, I^+ (respectively, I^-) denotes the set of points of I reached by future-directed (respectively, past-directed) timelike curves.

Brownian Motion of Coupled Nonlinear Oscillators: Thermalized Solitons and Nonlinear Response to External Forces

S. E. Trullinger
Department of Physics, University of Southern California, Los Angeles, California 90007

and

M. D. Miller and R. A. Guyer
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003

and

A. R. Bishop
Department of Physics, Queen Mary College, London E1 4NS, England

and

F. Palmer and J. A. Krumhansl
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

(Received 24 August 1977)

We study the steady-state dynamical behavior of a set of torsion-coupled pendula in the presence of damping, fluctuating thermal torques, and constant applied torque. For small applied torque, the average angular velocity of the pendula at low temperature is associated with the motion of thermalized sine-Gordon solitons and as the torque is increased the velocity response becomes strongly nonlinear. These results can be used to describe nonlinear response in Josephson transmission lines and weakly pinned one-dimentional charge-density–wave condensates.

Recently there has been growing interest in condensed-matter systems characterized by nonlinear wave equations which possess solitary-wave or soliton solutions.[1-8] The equilibrium statistical mechanics of these systems has proven to be very interesting.[6,9,10] In this Letter we

