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Nonlinear Diffusion Problem Arising in Plasma Physics

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(Received 17 March 1978)

In earlier studies of plasma diffusion with Okuda-Dawson scaling ($D \sim n^{-1/2}$), perturbation theory indicated that arbitrary initial data should evolve rapidly toward the separable solution of the relevant nonlinear diffusion equation. Now a Lyapunov functional has been found which is strictly decreasing in time and bounded below. The rigorous proof that arbitrary initial data evolve toward the separable solution is summarized. Rigorous bounds on the decay time are also presented.

Anomalous diffusion of hydrogen plasma across a purely poloidal octupole magnetic field has been observed experimentally[1] for densities $n_e \sim 5\times10^9$ cm^{-3} and poloidal fields B_p in the range 250 G–1 kG. The diffusion coefficient for this anomalous diffusion scales like $D \sim n^{-1/2}$ with D being independent of B_p. This scaling was predicted for vortex diffusion by Okuda and Dawson[2] when the ratio of ion plasma frequency and ion cyclotron frequency satisfies $\Omega_{pi}^2/\Omega_{ci}^2 \gg 1$, and was first observed by Tamano, Prater, and Ohkawa.[3] The Wisconsin octupole experiments have *shown that after a few milliseconds the density profile evolves into a fixed shape (the "normal mode") which then decays in time.* Numerical studies[1] indicated that the normal mode shape was well approximated by the shape of the separable solution of the relevant one-dimensional diffusion equation.

In normalized units, the diffusion equation may be written as[4,5]

$$\frac{\partial}{\partial x}\left[D(n)\frac{\partial n}{\partial x}\right] = F(x)\frac{\partial n}{\partial t} \text{ for } 0 \leq x \leq 1, \tag{1}$$

where x is the spatial variable corresponding to magnetic field potential and t is the time. The geometrical factor $F(x)$ is a positive function determined by the octupole geometry and in general $D(n) \propto n^{\delta}$ where $\delta \geq -1$. It is convenient to rewrite this equation as

$$u_{xx} = F(x)(u^{q-1})_t, \tag{2}$$

where $q=(2+\delta)/(1+\delta)$ and $n=u^{q-1}$. By use of a perturbation analysis, it was established that the separable solution of (2) is stable against small perturbations. If the separable solution is written as $u(x,t)=S(x)T(t)$, then for $F(x)=$const all perturbations decay at least as fast as $T^4(t)$.

In this Letter, we report the first rigorous results on this novel nonlinear diffusion problem. Methods have now been developed to prove that an arbitrary initial density profile will evolve toward the separable solution for $2<q<\infty$ $(0>\delta>-1)$. To save space, the methods will be illustrated on the model diffusion equation

$$u_{xx}(x,t) = 2u(x,t)u_t(x,t), \tag{3}$$

with $u(0,t)=u(1,t)=0$, since this corresponds to one case of experimental interest. All the arguments presented here may be generalized for all $2<q<\infty$. When $F(x)=$const, the generalization enjoys full mathematical rigor.[6] When $F(x)$ is an arbitrary positive integrable function, the generalization may at least be carried through formally.[7]

Sabinina[8] has established the existence and uniqueness of the solutions of (3). She has also shown that a finite time t^* exists when the solution first vanishes identically. We call this time t^* the extinction time.

The proof that arbitrary initial data evolve toward the separable solution will now be summarized. First, introduce the function

$$w(x,t) = u(x,t)/(t^* - t), \tag{4}$$

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and the functional

$$I(w)=\tfrac{1}{2}\int w_x{}^2dx-\tfrac{2}{3}\int w^3dx. \tag{5}$$

We wish to prove that w tends toward the nontrivial positive function w_0 that satisfies

$$w_{xx}+2w^2=0, \tag{6}$$

since this is one form of the equation for the shape function $S(x)$ of the separable solution. Substituting (4) into (3) and defining a new time coordinate by $\tau=-\ln(1-t/t^*)$, we find that w satisfies

$$w_{xx}+2w^2=2ww_\tau. \tag{7}$$

Taking the τ derivative of (5), integrating once by parts, and using (7), we find

$$\frac{d}{d\tau}I=-2\int ww_\tau{}^2dx\leq 0. \tag{8}$$

Since w is positive for all $\tau<\infty$ and $w_\tau=0$ only when w satisfies (6), $dI/d\tau<0$ for all $\tau<\infty$ unless w already solves (6). Clearly I is bounded above. If I is bounded below, w must tend toward a limiting function as we will show.

To prove that I is bounded below, consider the integral

$$\int u_x{}^2\,dx=-\int u_{xx}u\,dx=-\int\frac{u_{xx}}{u^{1/2}}u^{3/2}\,dx. \tag{9}$$

Using Schwarz's inequality, we obtain

$$\left(\int u_x{}^2\,dx\right)^2\leq\int\frac{u_{xx}{}^2}{u}\,dx\int u^3\,dx. \tag{10}$$

Now define $Q(t)=\int u^3(x,t)\,dx$ and $R(t)=\int u_x{}^2(x,t)\,dx$. Then it is not difficult to show using (3) that

$$\frac{d}{dt}Q=-\tfrac{2}{3}R \text{ and } \frac{d}{dt}R=-\int\frac{u_{xx}{}^2}{u}\,dx. \tag{11}$$

Substituting (11) into (10), rearranging terms, integrating once, exponentiating, and again rearranging, we find

$$\frac{d}{dt}Q^{1/3}(t)\geq-\tfrac{1}{2}R(0)Q^{-2/3}(0). \tag{12}$$

Integrating (12) from 0 to t^* and using the definition of t^*, we obtain

$$t^*-t\geq 2Q(t)R^{-1}(t). \tag{13}$$

Using similar manipulations again on (13), we finally obtain $Q(t)\leq Q(0)(1-t/t^*)^3$ or equivalently

$$\int w^3(x,\tau)\,dx\leq\int w^3(x,0)\,dx. \tag{14}$$

Thus, $\int w^3\,dx$ is bounded above and $I(w)$ is correspondingly bounded below.

Since (6) is satisfied by $w=0$ as well as by the separable solution w_0, we must exclude the possibility that $w\to 0$. Using elementary calculus and Schwarz's inequality, we find that

$$u^2(x,t)\leq x(1-x)R(t) \tag{15}$$

from which it follows easily that

$$Q^{2/3}(t)\leq[\mathrm{B}(\tfrac{5}{2},\tfrac{5}{2})]^{2/3}R(t), \tag{16}$$

where $\mathrm{B}(x,y)$ is the beta function. Using (11) and the same type of manipulations used to deduce (14), we find

$$t^*-t\leq 2\left(\frac{3\pi}{2^7}\right)^{2/3}Q^{1/3}(t)\leq 0.351\max u(x,t), \tag{17}$$

from which it follows that $2.85\leq\max w(x,\tau)$ for all τ. Thus, the possibility that $w\to 0$ as $t\to t^*$ has been excluded.

The remainder of the argument involves a number of technical details[6] which need not concern us here. The argument may be summarized as follows: If w satisfies (6), we are done. If $w\neq w_0$, then $I(w)$ is a strictly decreasing function of τ which is bounded below by $I(w_0)$. Then for some sequence of times $(\tau_n\to\infty)$, $\lim\int ww_\tau{}^2\,dx\to 0$ and, hence, $\lim w\to w_0$. Some additional work is required to establish the uniformity with which w approaches w_0.[6] *Thus, the desired result that arbitrary initial data evolve toward the separable solution has been established.*

Another interesting aspect of this problem is the determination of t^*.[9] One rigorous upper bound on t^* in terms of the initial data may be obtained from (17). We find

$$t_u=\kappa Q^{1/3}(0)\geq t^*, \tag{18}$$

where $\kappa=0.351$ in this case. A more refined argument,[7] based on the fact that $d^2Q^{1/3}(t)/dt^2\geq 0$ while $dQ^{1/3}(t)/dt\leq-\tfrac{1}{2}\lambda c^{1/3}$ where $\lambda=11.7967$ and $c=\int S^3\,dx=0.4$ (see Ref. 5 for notation), gives $\kappa=0.2301$. This bound is the best possible one of this type since equality is achieved when $w=w_0$.

One rigorous lower bound is given by (13). Another is obtained by considering the function $S(x)=w_0(x)/w_0(\tfrac{1}{2})$ where $w_0(\tfrac{1}{2})=\tfrac{1}{2}\lambda$. Then define

$$ca_0(t)=\int u(x,t)S^2(x)\,dx \tag{19}$$

and

$$cB(t)=\int u^2(x,t)S(x)\,dx. \tag{20}$$

From (3) and Schwarz's inequality, we find

$$dB(t)/dt=-\lambda a_0(t)\geq-\lambda B^{1/2}(t). \tag{21}$$

TABLE I. Values of the rigorous lower bound t_L, perturbation estimate t_p, extinction time t^*, and rigorous upper bound t_u for five numerical experiments on Eq. (3). The initial distributions for the first four cases were

$$u(x,0)=\sum_{l=0}^{3} a_l \sin(l+1)\pi x$$

with (a_1, a_2, a_3, a_4) equal to (i) (1, 0.4, 0, 0), (ii) (1, 0, 0.3, 0), (iii) (1, 0, −0.3, 0), and (iv) (1, 0, 0, 0.225). The fifth case was for constant initial data $u(x,0)=1$.

Case	t_L	t_p	t^*	t_u
i	0.1834	0.1837[a]	0.1847	0.1927
ii	0.1673	0.1675	0.1677	0.1750
iii	0.1894	0.1894[a]	0.1895	0.1925
iv	0.1761	0.1762	0.1762	0.1793
v	0.2102	0.2104	0.2107	0.2301

[a]Ref. 11.

(This idea was suggested to us by Varadhan.[10]) Integrating (21) we find the rigorous lower bound

$$t_L=\frac{2}{\lambda}B^{1/2}(0)\leq t^*. \tag{22}$$

In Ref. 5, an estimate of t^* was obtained using a perturbation analysis. That estimate was

$$t_p=\frac{B(0)+a_0^{\,2}(0)}{\lambda a_0(0)}\approx t^*. \tag{23}$$

Using the fact that $2\leq y+y^{-1}$ for $y=B^{1/2}/a_0$, we find immediately that $t_L\leq t_p$. Thus, the perturbation estimate is always greater than or equal to the rigorous lower bound (22). By expanding u in terms of the eigenfunctions of Ref. 5, we can also show that $Q^{1/3}(t)\cong c^{1/3}B(t)/a_0(t)$. It follows that $t_u\approx 2B(0)/\lambda a_0(0)\geq t_p$.

In Table I, we compare our rigorous bounds and the perturbation estimate t_p to the value of t^* observed in numerical experiments. The first four cases are the same as those in Ref. 5. The fifth case is for constant initial data $u(x,0)=1$. As anticipated, these computer experiments (and many others) show that $t_L\leq t_p\leq t^*\leq t_u$; however, the relation $t_p\leq t^*$ is not universal.

It has recently been observed experimentally that, in the presence of a weak toroidal field ($B_T/B_p\gtrsim 0.1$), the enhanced vortices required for Okuda-Dawson diffusion are damped away in a collisional plasma.[12] The plasma diffusion was observed to be classical ($\delta=1$) for all values of B_p studied ($B_p\lesssim 1.25$ kG). *However, "normal mode" behavior persisted as predicted by the perturbation analysis of Refs. 4 and 5.* This observation provides a second experimental confirmation of the correctness of our analysis. Furthermore, it suggests that such behavior may be a very general property of enclosed, nonlinear, diffusive systems.

This research was supported in part by the National Science Foundation, Grant No. NSF-MCS-75-09837, in part by a grant from the Sloan Foundation, and in part by the U. S. Air Force Office of Scientific Research, Grant No. AFOSR-77-3286.

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