New Formalism for High-Energy Scattering

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Starting from an operator eikonal formula, we develop a new formalism which enables us to prove that, in quantum electrodynamics, a particle becomes completely absorptive at high energies.

We recently developed a new scheme for unitarizing S-matrix amplitudes in gauge field theories in the high-energy limit with fixed momentum transfer. We have (i) established a consistent procedure for determining which Feynman diagrams and which contributions from these diagrams to include in calculating the scattering amplitudes, and (ii) found that the sum of the resulting amplitudes is given by an eikonal form which is explicitly unitary and which is valid for all processes, both elastic and inelastic. In outline, we used the unitarity condition and crossing symmetry to determine which set of Feynman diagrams to calculate. This is possible because the unitarity condition interrelates the amplitudes of different Feynman diagrams. In particular, if we include a set of lower-order Feynman diagrams, we must then also include all higher-order diagrams which are related to the lower-order ones by unitarity. Starting with the lowest-order amplitudes for all elastic and inelastic reactions (with the propagators in the Yang-Mills theory Reggeized), a unique set of Feynman diagrams is thereby generated. The amplitudes of these diagrams are then computed using Feynman rules.[1] In quantum electrodynamics (QED)[2] we calculated the appropriate diagrams for all processes and to all orders of perturbation theory. In Yang-Mills theory[3] we calculated the elastic scattering amplitude through the tenth order in perturbation theory. In each case, the two-body elastic-scattering amplitude in the impact-distance space is given by the eikonal form

$$S(\vec{b}, T) = \langle 0|\exp[i\chi(\vec{b}, T)]|0\rangle. \qquad (1)$$

In (1), $\vec{b}$ is the two-dimensional vector representing the impact distance between the two incident particles, $T = \ln s/2\pi \gg 1$, where s is the square of the center-of-mass energy of the system, $|0\rangle$ is the state containing the two colliding particles (i.e., no created particles), and the eikonal $\chi(\vec{b}, T)$ is a Hermitian operator representing the sum of the lowest-order amplitudes in all reactions, with the propagators in the Yang-Mills case Reggeized. Similar eikonal forms have been proposed before as models for high-energy scattering.[4] In contrast we have obtained it not by assumption but by a systematic program of calculation in field theory.

Unlike the eikonal formula in potential scattering, to which it bears a formal resemblance, Eq. (1) is not yet in a form from which physical consequences can be readily extracted. Because it must describe creation and annihilation of particles occurring in hadron scattering, the eikonal χ is an operator. As a result, the matrix elements of $e^{i\chi}$ are not related to those of χ in a simple manner, and the physical implications of the eikonal formula (1) remain to be deduced.

The expressions for the eikonal χ in QED and the Yang-Mills theory are fairly complicated. In the QED case, the complication is mostly due to the fact that a created electron must be accompanied by a created positron, while in the Yang-Mills case the complication is due to the complexity of the vertex factors and the Reggeization of the propagators. As a first attempt to understand the consequences of eikonalization, we shall consider the simplified model in which a particle can be created singly with the vertex factor equal to the coupling constant g, and the propagator in the momentum space is simply $(\vec{q}_\perp^{\,2}+\lambda^2)^{-1}$, where $\vec{q}_\perp$ and λ are, respectively, the transverse momentum and the mass of the virtual particle. For this model, we have

$$\chi(\vec{b}, T) = g^2\Big[K(b) + g\int d^2b_1\int_0^T dT_1 K(|\vec{b}-\vec{b}_1|)\,x(\vec{b}_1, T_1)K(b_1) + \cdots + g^n\int \prod_{i=1}^{n} d^2b_i \int_0^T dT_1\int_0^{T_1}dT_2\cdots\int_0^{T_{n-1}}dT_n K(|\vec{b}-\vec{b}_1|)x(\vec{b}_1, T_1)K(|\vec{b}_1-\vec{b}_2|)x(\vec{b}_2, T_2)\cdots K(b_n) + \cdots\Big]. \qquad (2)$$

In (2), b equals $|\vec{b}|$, $K(b) = K_0(\lambda b)/2\pi$ is the Fourier transform of $(\vec{q}_\perp^{\,2}+\lambda^2)^{-1}$ (the meson propagator in the impact-distance space), and

$$x(\vec{b}_i, T_i) = [a^\dagger(\vec{b}_i, T_i) + a(\vec{b}_i, T_i)]/\sqrt{2},$$

with $a^\dagger(\vec{b}_i, T_i)$ and $a(\vec{b}_i, T_i)$ the operators for creating and annihilating a particle at $\vec{b}_i$ with rapidity (in the laboratory system) $2\pi T_i$. The creation and annihilation operators satisfy the commutation rule

$$[a(\vec{b}_i, T_i), a^\dagger(\vec{b}_j, T_j)] = \delta(T_i - T_j)\delta^{(2)}(\vec{b}_i - \vec{b}_j).$$

Thus the first term in (2) represents the elastic contribution, the second term the contribution from the creation or annihilation of a single particle, and so on.

As an artifice to facilitate the calculation of $S(\vec{b}, T)$ we shall divide the $\vec{b}$ and T spaces into small regions and approximate the integrals in (2) by sums. (Eventually, we recover these integrals by going to the limit when these small regions are infinitesimal.) Thus we replace the b space by a two-dimensional lattice with the lattice constant (distance between two neighboring lattice points) d, and replace the T space by a one-dimensional lattice with the lattice constant ϵ. We shall use the index j (two components implied) to denote a lattice point in the b space, and the index n to denote a lattice point in the T space. We shall also replace the creation and annihilation operators by $a_j^\dagger(n)$ and $a_j(n)$, which satisfy the commutation rule $[a_i(n), a_j^\dagger(m)] = \delta_{ij}\delta_{nm}$.

With such replacements, and some algebraic manipulation, the eikonal in (2), denoted here by $\chi_j(N)$, can be shown to be equal to the jth component of a vector $\vec{\chi}(N)$, where

$$\vec{\chi}(N) = [I + 2\sqrt{\epsilon}\,\Lambda x(N)][I + 2\sqrt{\epsilon}\,\Lambda x(N-1)]\cdots[I + 2\sqrt{\epsilon}\,\Lambda x(1)]\vec{\chi}(0). \quad (3)$$

In (3), $N\epsilon = T$, I is the identity matrix, and $\vec{\chi}(N)$ is a vector which has a component associated with each lattice point in the b space, with $[\chi(0)]_j = g^2K(|\vec{b}_j|)$. Also, Λ and $x(n)$ in (3) are matrices with the matrix elements $\Lambda_{ij} = dgK(|\vec{b}_i - \vec{b}_j|)/2$, $[x(n)]_{ij} = x_i(n)\delta_{ij}$ associated with each pair of lattice points i and j, where $x_i(n) = [a_i(n) + a_i^\dagger(n)]/\sqrt{2}$.

Since the operators $x_j(n)$ commute with one another, an eigenstate of $\chi_i(n)$ is a product of the eigenstates of the operators $x_j(n)$ involved in (3). In other words, each eigenstate of $\chi_i(N)$ is specified by a designation of the quantum numbers of $x_j(n)$ for all j and for all positive n less than or equal to N.

It is therefore possible to calculate $S(\vec{b}, T)$ in (1), denoted by $S_j(N)$ below, by expanding the ground state $|0\rangle$ into a superposition of the eigenstates of $\chi_j(N)$. Since the ground-state wave function of a harmonic oscillator is $\exp(-\frac{1}{2}x^2)/\pi^{1/4}$, we have from (1) and (3) that

$$S_j(N) = \int \prod_{1\le n\le N} \prod_j \{dx_j(n)\exp[-x_j^2(n)]/\sqrt{\pi}\}$$
$$\times \exp i\{[I + 2\sqrt{\epsilon}\,\Lambda x(N)][I + 2\sqrt{\epsilon}\,\Lambda x(N-1)]\cdots[I + 2\sqrt{\epsilon}\,\Lambda x(1)]\vec{\chi}(0)\}_j. \quad (4)$$

We may think of $x(n)$ in (4) as a diagonal matrix whose matrix elements $x_j(n)$ are random variables with Gaussian distributions. Thus $S_j(N)$ is equal to the expectation value of the exponential of the jth component of a random vector $i\vec{\chi}(N)$ which, by (3), is equal to a product of random matrices operating on $i\vec{\chi}(0)$. We shall study $S_j(N)$ in the limit $d\to 0$, $\epsilon\to 0$, with $T = \epsilon N \gg 1$.

The physical meaning of the eikonal form (4) is very suggestive. Let us imagine a three-dimensional lattice with its lattice points specified by the index (j,n). We may think of j as the index specifying the transverse position on the lattice and n the index specifying the longitudinal position on the lattice. While the transverse dimension of the lattice is infinite, the longitudinal dimension of the lattice is equal to $N\epsilon = T$. As $s = e^{2\pi T}$ increases, the longitudinal dimension of the lattice also increases.

There is, associated with each lattice point (j,n), a harmonic oscillator with the creation operator $a_j^\dagger(n)$ and the annihilation operator $a_j(n)$. When two high-energy particles collide, they can excite any of these harmonic oscillators associated with the three-dimensional lattice in any arbitrary manner. The scattering is therefore a stochastic process in which quanta of the harmonic oscillators are created and annihilated in a random way. It is interesting to observe that the relevant physical entity which directly enters is not the creation operator or the annihilation operator separately, but the combination $x = (a + a^\dagger)/\sqrt{2}$. The eigenvalues of x play the role of a random variable which can take any value between $-\infty$ and ∞, with the probability distribution equal to the Gaussian $\exp(-x^2)/\sqrt{\pi}$. It is also important to observe that the random variables $x_j(n)$ enter in the form of a power series for the eikonal χ, not for the S matrix. As s becomes larger and larger, the three-dimensional lattice expands in the longitudinal direction, and more and more harmonic oscillators are involved. Thus χ receives contributions from an increasing number of random variables as s increases. Consequently, the expectation value of $\langle 0|\chi_j^2(N)|0\rangle$, for example, is very large as $s\to\infty$. Indeed, in the QED case, this expectation value corresponds to the sum of tower diagrams[2] and violates the Froissart bound. It is important that the S matrix $S_j(N)$, being equal to $\langle 0|\exp[i\chi_j(N)]|0\rangle$, always satisfies unitarity no matter how large $\chi_j(N)$ becomes. Indeed, let $\rho_{j,N}(\chi)$ be the probability that the eigenvalue of $\chi_j(N)$ is equal to χ; then we have

$$S_j(N)=\int_{-\infty}^{\infty}d\chi\rho_{j,N}(\chi)e^{i\chi}.$$

If $\rho_{j,N}(\chi)$ is concentrated in the region where χ is very large, we expect that the rapid oscillation of the integrand makes the integral above vanish.

We shall show below that this is indeed the case. Let us define $\vec{\xi}$ to be a vector which has a component associated with each lattice point of the b space, and

$$S(\vec{\xi},N\epsilon)\equiv\int\prod_{n=1}^{N}\prod_{j}\{dx_j(n)\exp[-x_j^2(n)]/\sqrt{\pi}\}\exp[i\vec{\chi}(N)\cdot\vec{\xi}]. \tag{5}$$

If we set $\xi_j=1$ for a certain j, and set all other components of $\vec{\xi}$ to zero, then $S(\vec{\xi},N\epsilon)$ is equal to $S_j(N)$ given by (4). Thus, a knowledge of $S(\vec{\xi},T)$ contains more than the complete information for the S matrix over the whole b lattice.

Equation (3) gives the recursion formula $\vec{\chi}(N+1)=[I+2\sqrt{\epsilon}\Lambda x(N+1)]\vec{\chi}(N)$. By making use of this and taking the desired limit $\epsilon\to 0$, it is straightforward to show that

$$\partial S(\vec{\xi},T)/\partial T=HS(\vec{\xi},T), \tag{6}$$

where

$$H=\sum_j(\sum_i\Lambda_{i-j}\xi_i)^2(\partial^2/\partial\xi_j^2) \tag{7}$$

with the initial condition $S(\vec{\xi},0)=\exp[i\vec{\xi}\cdot\vec{\chi}(0)]$.

A standard way to analyze the partial differential equation above is to perform a Laplace transform with respect to T. Let us define

$$\tilde{S}(\vec{\xi},w)\equiv\int_0^\infty dT\,e^{-wT}S(\vec{\xi},T), \tag{8}$$

then

$$S(\vec{\xi},T)=\int_{\delta-i\infty}^{\delta+i\infty}\frac{dw}{2wi}e^{wT}\tilde{S}(\vec{\xi},w), \tag{9}$$

where δ is any positive number. Then

$$\tilde{S}(\vec{\xi},w)=-(H-w)^{-1}\exp[i\vec{\xi}\cdot\vec{\chi}(0)].$$

We shall prove that, in the desired limit $d\to 0$, $\tilde{S}(\vec{\xi},w)$ is an entire function of w. Thus we can set $\delta=0$ in (9) and w becomes purely imaginary on the entire contour of integration. Consequently, as $T\to\infty$, the integrand in (9) oscillates rapidly, and $S(\vec{\xi},T)$ vanishes as $T\to\infty$ with $\vec{\xi}$ fixed.

To prove that $\tilde{S}(\vec{\xi},w)$ is entire in the limit $d\to 0$, let us note that Λ is proportional to d. Thus, in the operator H as given by (7), we may set the diagonal matrix element Λ_{jj} to zero. This is because we can always drop one infinitesimally small term from a sum which is an approximation of an integral.

The key observation is that once we set Λ_{jj} to zero, the operator H is self-adjoint and negative definite (or nonpositive). The latter property follows from

$$\int d^3\xi\varphi^*(\vec{\xi})H\varphi(\vec{\xi})=-\int d^3\xi\sum_j(\sum_i\xi_i\Lambda_{i-j})^2\left|\frac{\partial\varphi(\vec{\xi})}{\partial\xi_j}\right|^2\leq 0. \tag{10}$$

The operator H is also invariant under scale transformations $\xi_j\to c\xi_j$, all j. Hence it is helpful to introduce the spherical coordinates $r\equiv(\sum\xi_j^2)^{1/2}$, $\hat{\xi}_j\equiv\xi_j/r$. The operator H operating on $r^{-\eta}$ times a function of $\hat{\xi}_j$ is always equal to $r^{-\eta}$ times another function of $\hat{\xi}_j$. We may therefore define the operator $\mathcal{K}(\eta)$ by $Hr^{-\eta}F(\hat{\xi}_j)\equiv r^{-\eta}\mathcal{K}(\eta)F(\hat{\xi}_j)$, where $\mathcal{K}(\eta)$ involves only the angular variables $\hat{\xi}_j$. If we express $\exp[i\vec{\xi}\cdot\vec{\chi}(0)]$ by its Mellin transform integral

$$e^{i\vec{\xi}\cdot\vec{\chi}(0)}=\int_{L-i\infty}^{L+i\infty}\frac{d\eta}{2\pi i}[r\hat{\xi}\cdot\vec{\chi}(0)]^{-\eta}e^{i\pi\eta/2}\Gamma(\eta),$$

where L is any positive constant, then

$$\tilde{S}(\vec{\xi},w)=-\int_{L-i\infty}^{L+i\infty}\frac{d\eta}{2\pi i}r^{-\eta}e^{i\pi\eta/2}\Gamma(\eta)[\mathcal{K}(\eta)-w]^{-1}[\hat{\xi}\cdot\vec{\chi}(0)]^{-\eta}. \tag{11}$$

Let us for the moment consider the case in which the lattice b space has M lattice points. Then it can be proved that, since H is self-adjoint, we have $\mathcal{K}(\eta)=\mathcal{K}^\dagger(M-\eta^*)$. In particular, the above equation states that $\mathcal{H}(\eta)$ is Hermitian on the line $\mathrm{Re}\,\eta = M/2$. We therefore move the contour of integration in (11) to this line, and get

$$\tilde{S}(\vec{\xi},w) = -\int_{-\infty}^{\infty}\frac{dp}{2\pi}(re^{-i\pi/2})^{-(M/2)-ip}\,\Gamma\left(\frac{M}{2}+ip\right)\left[\mathcal{H}\left(\frac{M}{2}+ip\right)-w\right]^{-1}[\hat{\xi}\cdot\vec{\chi}(0)]^{-(M/2)-ip}. \quad (12)$$

From (12), it is seen that $\tilde{S}(\vec{\xi},w)$ has singularities at the points of w equal to the eigenvalues of $\mathcal{H}(\frac{1}{2}M+ip)$. Since all of these eigenvalues are negative, $\tilde{S}(\vec{\xi},w)$ has singularities on the negative real axis of w.

As $M\to\infty$, all of these eigenvalues become negatively infinite. This is because they are also the eigenvalues of H corresponding to the eigenfunctions in the form of $r^{-(M/2)-ip}$ times a function of $\hat{\xi}_j$. The radial derivatives of these eigenfunctions are equal to infinity as $M\to\infty$. Thus, from (10), the corresponding eigenvalues go to $-\infty$ as $M\to\infty$. Consequently, $\tilde{S}(\vec{\xi},w)$ has no singularities in the finite w plane. In other words, $\tilde{S}(\vec{\xi},w)$ is entire.

The considerations above can be directly extended to the eikonal formula in QED. In this case, we have,[5] instead of (7),

$$H = \sum_{ijkl}\xi_i\xi_j K_{ijkl}\frac{\partial^2}{\partial\xi_k\partial\xi_l}, \quad (13)$$

where K_{ijkl} is proportional to d^4. In the limit $d\to 0$, H is again Hermitian, with all its eigenvalues negatively infinite. Thus $\tilde{S}(\vec{\xi},w)$ is again an entire function of w, and $S(\vec{\xi},T)$ vanishes in the limit $T\to\infty$ with $\vec{\xi}$ fixed. Physically, this means that a target particle becomes completely absorptive at high energies.

We have just begun to explore the consequences of the eikonal formula, and many important problems remain to be solved. We list some of them here: (i) Although we have shown that $S(\vec{\xi},T)$ vanishes as $T\to\infty$, the rate at which it vanishes is still unknown. This will be of experimental interest. (ii) We have studied only the elastic-scattering amplitude. The eikonal formula is applicable to all reactions, elastic or inelastic. Extension of the treatment to inelastic amplitudes will enable us to determine the behavior of inelastic cross sections, the multiplicity, the inclusive distributions, etc. (iii) The formalism we use has been applied only to QED. The consequences of the eikonal formula for the Yang-Mills case are yet to be deduced. In particular, since the vector-meson propagators in the Yang-Mills case are Reggeized, all matrix elements of the eikonal operator vanish at infinite energy. It is therefore not at all clear that the S matrix would approach zero as the energy approaches infinity. More generally, it will be important to determine if the physical consequences of Yang-Mills theories are qualitatively different from those in QED.

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[1]For a precise description of the approximations involved in this procedure, see H. Cheng, J. Dickinson, C. Y. Lo, K. Olaussen, and P. S. Young, to be published, and H. Cheng, J. Dickinson, C. Y. Lo, and K. Olaussen, to be published.

[2]Cheng, Dickinson, Lo, Olaussen, and Yeung, Ref. 1.

[3]Cheng, Dickinson, Lo, and Olaussen, Ref. 1.

[4]S. Auerbach, R. Aviv, R. Sugar, and R. Blankenbecler, Phys. Rev. D 6, 2216 (1972); R. Aviv, R. Sugar, and R. Blankenbecler, Phys. Rev. D 5, 3252 (1972). See also G. Calucci, R. Jengo, and C. Rebbi, Nuovo Cimento A 4, 330 (1971), and 6, 601 (1971).

[5]For details see H. Cheng, J. Dickinson, K. Olaussen, and P. S. Yeung, "Consequences of Eikonalization" (unpublished).