Hydrodynamic Stability of ^{3}He-A

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The stability of ^{3}He-A with uniform superflow is examined in the hydrodynamic limit. Below the threshold found by Bhattacharyya, Ho, and Mermin, any reasonable choice of hydrodynamic parameters renders the uniform texture stable with respect to general three-dimensional perturbations. Beyond threshold, the texture undergoes a transition to a static helix, whose broken translational symmetry complicates the general stability analysis. In a truncated variational approximation, this structure is stable near threshold with respect to perturbations proportional to $\exp(i\vec{k}\cdot\vec{r})$.

Recent studies of superfluid ^{3}He-A have revealed that uniform hydrodynamic flow can render the uniform texture unstable.[1-3] This possibility arises from the competition between the hydrodynamic torque that tends to bend $\hat{l}$ and the curvature energy that opposes such deformation. In particular, Bhattacharyya, Ho, and Mermin[1] have analyzed the hydrodynamic free energy to obtain a simple criterion for the stability of the uniform texture with respect to small deviations in the direction of $\hat{l}$. For bulk fluid, the dipole energy couples $\hat{d}$ and $\hat{l}$, in effect stiffening the system. With the weak-coupling Ginzburg-Landau parameters near T_c, this enhancement just suffices to stabilize the uniform texture. At lower temperature, however, the dipole-locked uniform configuration may become unstable because of the decreased anisotropy ρ_0 in the superfluid-density tensor. The present paper examines the character of the instability and the resulting deformed state in the hydrodynamic approximation.[4-6] Just beyond threshold at ρ_{0c}, the uniform $\hat{l}$ vector undergoes a second-order "displacive" transition to a stable helical configuration. The vector $\hat{l}$ then has a maximum apex angle proportional to $(\rho_{0c}-\rho_0)^{1/2}$ and a wave number proportional to the superfluid velocity. This behavior typifies a Landau-type transition.

My analysis starts from the hydrodynamic free-energy density[7,8]

$$f=\tfrac{1}{2}\rho_s v_s^2-\tfrac{1}{2}\rho_0(\hat{l}\cdot\vec{v}_s)^2+c\vec{v}_s\cdot\operatorname{curl}\hat{l}-c_0(\vec{v}_s\cdot\hat{l})(\hat{l}\cdot\operatorname{curl}\hat{l})+\tfrac{1}{2}K_s(\operatorname{div}\hat{l})^2+\tfrac{1}{2}K_t(\hat{l}\cdot\operatorname{curl}\hat{l})^2+\tfrac{1}{2}K_b(\hat{l}\times\operatorname{curl}\hat{l})^2 \tag{1}$$

and the corresponding mass-current density

$$\vec{j}_s=\partial f/\partial\vec{v}_s=\rho_s\vec{v}_s-\rho_0\hat{l}(\hat{l}\cdot\vec{v}_s)+c\operatorname{curl}\hat{l}-c_0\hat{l}(\hat{l}\cdot\operatorname{curl}\hat{l}). \tag{2}$$

I follow Bhattacharyya, Ho, and Mermin and take $\vec{v}_n=0$, which may be considered to arise from the presence of distant stationary walls. In the hydrodynamic limit, the orbital part of the order parameter of ^{3}He-A is specified by the orientation of a rigid triad of orthonormal vectors, and the corresponding dynamics is obtained from three independent variations: $\delta\Phi$ characterizing a rotation about the unit vector $\hat{l}$, and $\delta\hat{l}$ characterizing the change in the direction of $\hat{l}$. These independent virtual displacements induce an associated change in the superfluid velocity,[4-6,9]

$$\delta\vec{v}_s=\nabla\delta\Phi-\hat{l}\times(\nabla\hat{l})\cdot\delta\hat{l}. \tag{3}$$

A straightforward analysis provides the coupled dynamical equations

$$\nabla\cdot\vec{j}_s=0 \tag{4}$$

$$\mu\frac{\partial\hat{l}}{\partial t}=\nabla_i\frac{\partial f}{\partial\nabla_i\hat{l}}-\frac{\partial f}{\partial\hat{l}}+\hat{l}\times(\vec{j}_s\cdot\nabla)\hat{l}, \tag{5}$$

where μ is the orbital viscosity and the partial derivatives of f are taken at fixed $\vec{v}_s$.

Before treating the helical configuration, it is helpful to review the stability of the uniform state $\hat{l}=\hat{z}$ and $\vec{v}=w\hat{z}$. Small deformations of this texture may be characterized by the phase variable $\delta\Phi(\vec{r},t)$ and a two-dimensional vector $\delta\hat{l}(\vec{r},t)$ confined to the xy plane. The translational invariance of the unperturbed configuration permits a complete solution of the linearized version of Eqs. (4) and (5) in the form of three-dimensional plane waves proportional to $\exp(i\vec{k}\cdot\vec{r}-\sigma t)$. It is convenient to use Eq. (4) to eliminate the amplitude $\delta\Phi_{\vec{k}}$, leading to a pair of equations for the two components of $\delta\hat{l}_{\vec{k}}$; these equations also follow directly from the free-energy density in Eq. (4) of Ref. 1. The time-decay parameter

σ satisfies a quadratic equation:

$$[\sigma\mu-\rho_0 w^2-K_b k_z^2-K_s q^2+\rho_0^2 w^2 q^2(\rho_s^{\parallel} k_z^2+\rho_s q^2)^{-1}][\sigma\mu-\rho_0 w^2-K_b k_z^2-K_t q^2+c_0^2 k_z^2 q^2(\rho_s^{\parallel} k_z^2+\rho_s q^2)^{-1}]$$
$$-k_z^2 w^2[2c_0+\rho_s^{\parallel}-\rho_0 c_0 q^2(\rho_s^{\parallel} k_z^2+\rho_s q^2)^{-1}]^2=0, \tag{6}$$

where $\vec{q}$ is the projection of $\vec{k}$ onto the xy plane.

Both roots of (6) are real, and stability requires that they be positive for all k. It is straightforward to show that the sum of the roots is positive if

$$4K_b+2(K_s+K_t)>(c_0^2/\rho_s^{\parallel})+(\rho_s^{\parallel}/c_0^2)(2K_b-K_s-K_t)^2,$$

which is satisfied for any reasonable hydrodynamic parameters. The product of the roots $P(\vec{k})$ also must be positive, and it is simplest to parametrize $\vec{k}$ by its magnitude k and its polar angle χ measured from the z axis. In this way, $P(k,\chi)$ becomes a quadratic form in k^2, with coefficients that depend on χ. Near the threshold at

$$\rho_{0c}\equiv(c_0+\tfrac{1}{2}\rho_s^{\parallel})^2 K_b^{-1}, \tag{7}$$

$P(k,\chi)$ has a local minimum at $|k|=(c_0+\frac{1}{2}\rho_s^{\parallel})w/K_b$ and $\chi=0$ whenever $K_s+K_t>\frac{1}{4}\rho_s^{\parallel}$, which is satisfied for all plausible choices of the hydrodynamic parameters. Numerical evaluation for all χ confirms the conclusion of Ref. 1 that the uniform texture first becomes unstable at ρ_{0c} with $\chi=0$ (namely $\vec{q}=0$) and $|k_z|=(c_0+\frac{1}{2}\rho_s^{\parallel})w/K_b$.

This instability signals the onset of a helical distortion, and it is now simpler to project $\hat{l}$ on a spherical polar basis, with $l_x=\sin\theta\cos\varphi$, $l_y=\sin\theta\sin\varphi$, $l_z=\cos\theta$. The basic dynamical equations (5) take the form

$$\mu\frac{\partial\theta}{\partial t}=\nabla\cdot\frac{\partial f}{\partial\nabla\theta}-\frac{\partial f}{\partial\theta}-\sin\theta\,\vec{j}_s\cdot\nabla\varphi, \tag{8a}$$

$$\mu\sin^2\theta\,\frac{\partial\varphi}{\partial t}=\nabla\cdot\frac{\partial f}{\partial\nabla\varphi}-\frac{\partial f}{\partial\varphi}+\sin\theta\,\vec{j}_s\cdot\nabla\theta, \tag{8b}$$

where the partial derivatives of f are again taken at constant $\vec{v}_s$. A static helix is characterized by an apex angle θ_0 and a wave number p, with the azimuthal angle given by $\varphi_0(z)=pz$. The corresponding superfluid velocity retains the general form $\vec{v}_s=w\hat{z}$, but the supercurrent now has a helical structure with

$$j_{sx}+i_{sy}=-\sin\theta_0 e^{ipz}[\rho_0 w\cos\theta_0+(c-c_0\sin^2\theta_0)p] \tag{9a}$$

and a uniform component along z,

$$j_{sz}=(\rho_s-\rho_0\cos^2\theta_0)w+c_0 p\sin^2\theta_0\cos\theta_0. \tag{9b}$$

This current satisfies Eq. (4). In addition, Eq. (8b) holds identically for a static helix, independent of p, and the remaining equation (8a), gives a condition for the equilibrium apex angle:

$$\sin\theta_0[j_{sz}p+\rho_0 w^2\cos\theta_0+c_0 wp(2-3\sin\theta_0)+2K_t p^2\cos\theta_0\sin^2\theta_0+K_b p^2\cos\theta_0(1-2\sin^2\theta_0)]=0. \tag{10a}$$

Any θ_0 and p satisfying this equation represent a possible equilibrium configuration, but they will not, in general, minimize the hydrodynamic free energy, given in Eq. (11) of Ref. 1. Imposing this additional condition readily yields a second relation,

$$p_{\rm opt}=-w\left[\frac{c_0\cos\theta_0+\frac{1}{2}(\rho_s^{\parallel}+\rho_0\sin^2\theta_0)\sec^2(\frac{1}{2}\theta_0)}{K_b\cos^2\theta_0+K_t\sin^2\theta_0+c_0\cos\theta_0(1-\cos\theta_0)}\right], \tag{10b}$$

which is the optimum value for a given set of hydrodynamic parameters.

The solution θ_0 of (10a) is just the uniform configuration considered previously. If, however, ρ_0 is less than the critical value ρ_{0c} and p is near the critical wave number

$$p_c\equiv-(c_0+\tfrac{1}{2}\rho_s^{\parallel})w/K_b, \tag{11}$$

the following second solution exists:

$$\theta_0^2 \approx \frac{K_b(c_0+\frac{1}{2}\rho_s^{\parallel})}{(2K_t-\frac{1}{2}\rho_s^{\parallel})(c_0+\frac{1}{2}\rho_s^{\parallel})-\frac{3}{2}\rho_s^{\parallel}K_b}\left[\frac{\rho_{0c}-\rho_0}{\rho_{0c}}-\left(\frac{p-p_c}{p_c}\right)^2\right]. \tag{12a}$$

The coefficient is positive for typical hydrodynamic parameters (it has the value $\frac{5}{4}$ for the weak-coupling Ginzburg-Landau parameters), and this relation therefore shows that small-angle helices can be in static equilibrium within the range of wave numbers

$$\left(\frac{p-p_c}{p_c}\right)^2 \leqslant \left(\frac{\rho_{0c}-\rho_0}{\rho_{0c}}\right) \ll 1. \tag{13}$$

Although the actual range of stability is narrower, we shall see that helices with $p \approx p_c$ are indeed stable near threshold. Similarly, an expansion of Eq. (10b) yields

$$p_{\text{opt}} \approx p_c\left[1-\theta_0^2\left(\frac{K_t+\frac{1}{2}c_0}{K_b}-\frac{1}{2}-\frac{\rho_0+\frac{3}{4}\rho_s^{\parallel}}{2c_0+\rho_s^{\parallel}}\right)\right], \tag{12b}$$

showing that p_c is the optimum value near threshold; the coefficient in the correction term has the value 7/60 for the same weak-coupling parameters.

To test for stability, we consider perturbations of the form

$$\theta=\theta_0+\delta\theta, \quad \varphi=\varphi_0+\delta\varphi, \quad \vec{v}_s=w\hat{z}+\delta\vec{v}_s, \tag{14}$$

where [see Eq. (3)]

$$\delta\vec{v}_s=\nabla\delta\Phi+p\sin\theta_0\delta\theta\hat{z}. \tag{15}$$

Equations (4) and (8) may be expanded to first order in the small quantities $\delta\Phi$, $\delta\theta$, $\delta\varphi$, and the translational invariance in the xy plane suggests a plane-wave structure $\exp(i\vec{q}\cdot\vec{r}_\perp-\sigma t)$, where $\vec{r}_\perp = x\hat{x}+y\hat{y}$. In contrast, the z dependence is complicated by the broken translational symmetry. For $q^2\neq 0$, simple axial plane waves $\exp(ik_z z)$ no longer satisfy the coupled equations because the periodic helical structure mixes in harmonics of the form $\exp[i(k_z+np)z]$, where n is an integer. It is not difficult to obtain the set of homogeneous algebraic equations for the associated amplitudes $\delta\Phi_n$, $\delta\theta_n$, $\delta\varphi_n$, but a full analysis is prohibitive. As an approximation, I have truncated them by including only the terms with $n=0$. To justify this procedure, we note that the instability of the uniform texture first appears with $q^2=0$, so that small q^2 is expected to remain the most important range. Furthermore, the exact coupled differential equations for $\delta\Phi$, $\delta\theta$, $\delta\varphi$ have a variational basis, equivalent to minimizing the free energy for general variations about the static helix. Since the terms that mix in the additional harmonic contributions vanish at $q^2=0$, pure plane waves $\exp(ik_z z)$ should provide a suitable trial basis for studying the long-wavelength behavior. Minimization within this restricted class of trial functions precisely reproduces the truncated equations with $n=0$.

In this approximation, the condition $\nabla\cdot\delta\vec{j}_s=0$ is readily solved for $\delta\Phi_0$, and substitution into the remaining equations yields a pair of algebraic homogeneous equations for $\delta\theta_0$ and $\delta\varphi_0$. As for the uniform texture, the decay constant σ for small-amplitude perturbations with $\vec{k}=\vec{q}+k_z\hat{z}$ satisfies a quadratic equation, and both roots are real. Again assuming $\rho_{0c}-\rho_0 \ll \rho_{0c}$ and $(p-p_c)^2 \ll p_c^2$, we find that the sum S and product P of the roots are given by

$$S=2[w^2(\rho_{0c}-\rho_0)-K_b(p-p_c)^2]+k_z^2[2K_b-\theta_0^2(2K_b-K_s-K_t)] \\ +q^2[K_s+K_t+\tfrac{1}{2}\theta_0^2(2K_b-K_s-K_t)]-c_0^2\theta_0^2(k_z^2-\tfrac{1}{2}q^2)^2(\rho_s^{\parallel}k_z^2+\rho_s q^2)^{-1}, \tag{16}$$

$$P=2k_z^2K_b[w^2(\rho_{0c}-\rho_0)-3K_b(p-p_c)^2]+q^2(K_s+K_t)[w^2(\rho_{0c}-\rho_0)-K_b(p-p_c)^2] \\ +[k_z^2K_b+\tfrac{1}{2}q^2(K_s+K_t)] \\ \times\{k_z^2K_b+\tfrac{1}{2}q^2(K_s+K_t)+\theta_0^2(k_z^2-\tfrac{1}{2}q^2)[K_s+K_t-2K_b-(k_z^2-\tfrac{1}{2}q^2)c_0^2(\rho_s^{\parallel}k_z^2+\rho_s q^2)^{-1}]\}. \tag{17}$$

For small θ_0^2 [see Eq. (12a)], both of these are positive definite for

$$\left(\frac{p-p_c}{p_c}\right)^2 \leqslant \frac{1}{3}\left(\frac{\rho_0-\rho_{0c}}{\rho_{0c}}\right)^2 \ll 1, \tag{18}$$

which ensures the existence of stable helices near threshold with wave number $\approx p_c$ and small apex angle. Since Eq. (18) is more restrictive than Eq. (13), the small-amplitude analysis is essential in determining the range of stability. Furthermore, a helix with wave number p that satisfies (13) yet violates (18) is unstable with respect to long-wavelength longitudinal deformations ($k_z \neq 0$, $\vec{q}=0$), but it remains stable for transverse deformations ($\vec{q}\neq 0$).

It is notable that the criterion for stability exhibits no threshold in the magnitude of the superfluid velocity w. For this reason, my analysis applies both to the "toroidal" configuration considered in Ref. 1 and to the uniform flow considered in Ref. 2. In the former case, w would decrease with increasing angle θ_0, according to Eq. (10) of Ref. 1, whereas w would be fixed in the latter case.

The present calculation suggests two important extensions. First, it will be interesting to include higher-harmonic contributions in the solutions for $\delta\Phi$, $\delta\theta$, $\delta\varphi$, such as those proportional to $\exp[i(k_z \pm p)z]$. The variational principle shows that this improved set of trial functions will improve the estimate for σ, but I do not anticipate a qualitative change in the stability criterion (18) near threshold.

A second important question is the region far from threshold. This problem involves a wider range of the hydrodynamic parameters, whose temperature dependence may be found for weak coupling from the work of Cross.[7] Near T_c, the uniform texture is stable in the dipole-locked limit. For $T \ll T_c$, however, the anisotropy term ρ_0 in the superfluid-density tensor decreases algebraically. When ρ_0 falls below ρ_{0c}, my analysis indicates that the texture acquires a stable helical structure with wave number p near p_c. Does this configuration remain stable for still lower temperatures, or does it itself become unstable? In the latter situation, it would be interesting to investigate whether the instability occurs for $\mathrm{Im}\,\sigma \neq 0$, which would indicate that the new state has intrinsic time dependence, and for $\vec{q}\neq 0$, which would suggest a nonzero vorticity $i\vec{q}\times\hat{z}p \times \sin\theta_0 \delta\theta$. Given the relation between vorticity transport and dissipation in superfluids,[1,9,10] the possibility of an instability with $\vec{q}\neq 0$ merits special attention.

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