# Is the Fermi Theory of Weak Interactions a Yang-Mills Theory in Disguise?

Fred Cooper
*Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545, and Institute for Advanced Study, Princeton, New Jersey 08540*

and

Gerald Guralnik
*Brown University, Providence, Rhode Island 02912*

and

Neal Snyderman
*Institute for Advanced Study, Princeton, New Jersey 08540*
(Received 9 February 1978)

We outline the proof of a consistent renormalization scheme for the Fermi theory $(\overline{\psi}\gamma_\mu\frac{1}{2}\tau\psi)^2$ in the context of a mean-field expansion in the collective excitation $A_i{}^\mu \equiv \overline{\psi}\gamma_\mu\frac{1}{2}\tau_i{}'\psi$. Ward's identities show that there is only one mass and coupling and that the induced vertices are identical to those of Yang-Mills theory with elementary gauge fields at low energy. The Green's-functions equations are displayed and used to derive an eigenvalue restriction on the coupling.

Despite the simplicity and accuracy of fit to weak interaction phenomonology of first-order four-fermion–type interactions it was necessary to discard them because of apparently insurmountable difficulties in calculating higher-order corrections because of divergences.

Starting from the work of Nambu and Jona-Lasinio[1] it has recently been suggested that these renormalizability problems for four-fermion interactions can be avoided.[2,3] That this is the fact for vector-vector- and scalar-scalar–type four-fermion interactions has been substantiated in great detail to all orders in the bound-state mean-field expansion.[2,4,5] Earlier analysis[3,6,7] showed that in lowest order many of the parameters of the theory were related as should be expected because of the small number of bare parameters in the Lagrangian. Subsequent analysis with[4,5] use of general methods has substantiated that the four-fermion models are characterized to all orders by relations among the renormalized parameters. As a by-product of this analysis, it was demonstrated that the mean-field renormalized four-fermion theories were identical to Yukawa-type theories with elementary bosons for a particular set of values of Yukawa-theory renormalized parameters related to the vanishing of the wave-function renormalization of the elementary bosons. We show here that there exists a consistent renormalization scheme for the Fermi theory with self-interaction $(\overline{\psi}\gamma_\mu\frac{1}{2}\tau\psi)^2$ in the context of resumming the theory in terms of the interaction of the collective excitation $A_i{}^\mu \equiv \overline{\psi}\gamma_\mu\frac{1}{2}\tau_i{}'\psi$ with itself and the original fermions. Because the collective excitation $A_\mu{}^i$ is the isospin current, it has universal coupling to all objects with isospin. This results in the renormalized theory written in terms of $A_\mu{}^i$, with $\psi$ having a Yang-Mills structure at low energy. Here we demonstrate to all orders in a bound-state loop expansion (i.e., a calculational scheme) that the renormalized vector-vector Fermi theory with isospin has the same induced vertices and renormalized coupling strengths between vector mesons as a Yang-Mills theory. We obtain the renormalized Schwinger-Dyson equations of the theory, which are quite simple because the form of the interaction is the analog of a $Z_3 = 0$ theory of elementary vector fields.

These results show there exists a scheme for calculating without a cutoff higher-order weak processes in theories of weak interactions based on four-fermion interactions. The details of such a calculation in the model of vector-vector with scalar-scalar interactions will be presented elsewhere.

The Lagrangian for the vector-vector SU(2) Fermi model can be written in the alternative form

$$L = \overline{\psi}(\gamma\cdot\partial - m_0)\psi + g_0\overline{\psi}\gamma_\mu\tfrac{1}{2}\tau\cdot A_\mu\psi - \tfrac{1}{2}g_0 A_\mu\cdot A_\mu + \overline{\eta}\psi + \overline{\psi}\eta + j_\mu\cdot A_\mu. \tag{1}$$

Since the composite–vector-meson field $A_\mu$ is the isospin current, it must couple universally to all objects carrying isospin including itself, which leads to the Yang-Mills structure of the induced vertices. We see this most easily by studying the isospin Ward identities. Under a constant isospin rotation, only the source terms in $L$ are noninvariant, and using the equation of motion we obtain the isospin Ward identity:

$$\partial_\mu(\bar{\psi}\gamma_\mu \tfrac{1}{2}\vec{\tau}\psi) = i(\bar{\eta}\tfrac{1}{2}\vec{\tau}\psi - \bar{\psi}\tfrac{1}{2}\vec{\tau}\eta + \vec{j}_\mu \times \vec{A}_\mu). \tag{2}$$

Defining $W$ by

$$\exp(iW) = \int dA_\mu d\psi d\bar{\psi} \exp(i\int L\, d^4x) \tag{3}$$

and the generator of one-particle irreducible (1PI) diagrams via

$$\Gamma = W - \int j_\mu A_{\mu\,\mathrm{cl}} - \int \bar{\eta}\psi_{\mathrm{cl}} - \int \bar{\psi}_{\mathrm{cl}}\eta, \tag{4}$$

one has, using the *field current identity* $\bar{\psi}\gamma_\mu\tau\psi \equiv A_\mu - j_\mu/g_0$, that

$$\partial_\mu\left(A_{\mathrm{cl}}{}^\mu + \frac{1}{g_0}\frac{\delta\Gamma}{\delta A_\mu}\right) = i\left(\bar{\psi}_{\mathrm{cl}}\frac{\tau}{2}\frac{\delta\Gamma}{\delta\bar{\psi}_{\mathrm{cl}}} - \frac{\delta\Gamma}{\delta\psi_{\mathrm{cl}}}\frac{\tau}{2}\psi_{\mathrm{cl}}\right) + A_\mu{}^{\mathrm{cl}} \times \frac{\delta\Gamma}{\delta A_{\mu\,\mathrm{cl}}}. \tag{5}$$

Defining the 1PI composite vector-boson–fermion vertex via

$$\Gamma_{\mu i}(123) = \frac{1}{g_0}\frac{\delta^3\Gamma}{\delta\psi(2)\delta\bar{\psi}(3)\delta A_{\mu i}(1)}, \tag{6}$$

one obtains from Eq. (5) a quantum-electrodynamics–like Ward identity,

$$\partial_\mu\Gamma_\mu(123) = S^{-1}(13)\delta(1-2)\tfrac{1}{2}\tau - \delta(1-3)\tfrac{1}{2}\tau S^{-1}(12), \tag{7}$$

where $S^{-1}$ is the full inverse propagator for the fermions.

From this equation we obtain $Z_1 = Z_2$, where in momentum space as $p, p+q \to 0$ one has

$$\Gamma_{\mu i}(p, p+q) + \frac{\tau_i}{2}\frac{\gamma_\mu}{Z_1}, \quad S^{-1}(p+q) \to \gamma\cdot(p+q)Z_2{}^{-1} + \tilde{M}(q^2=0). \tag{8}$$

Defining the renormalization of the $A_\mu$ field via

$$Z_3{}^{-1} = \left[\frac{d}{dq^2}[D^{-1}(q^2)]^{\mu\nu}\right]_{q^2=0}$$

where $D^{\mu\nu}$ is the full composite-vector-meson propagator, one has that the renormalized coupling of the gluons to the fermions is given by $g_r{}^2 = g_0{}^2 Z_2{}^2 Z_1{}^{-2} Z_3 = g_0{}^2 Z_3$. From Eq. (5) one can show that the low-energy behavior of the Fourier transform of the three-gluon vertex

$$\Gamma^{ijk}_{\mu\nu} = \frac{1}{g_0}\frac{\delta\Gamma}{\delta A^i_\mu(1)\,\delta A^j_\nu(2)\,\delta A^k_\lambda(3)}$$

is

$$\lim_{p, p+q\to 0} \Gamma^{\mu\nu\lambda}_{ijk}(p, p+q) \to \bar{Z}_1{}^{-1}\epsilon_{ijk}[g_{\mu\nu}(q-p)_\lambda + g_{\lambda\nu}(2p+q)_\mu - g_{\lambda\mu}(2q+p)_\nu] \tag{9}$$

and that

$$q_\mu\Gamma^{\mu\nu\lambda}_{ijk} = \epsilon_{ijk}[D^{-1}{}_{\nu\lambda}(p+q) - D^{-1}{}_{\nu\lambda}(p)]. \tag{10}$$

From this, one finds $\bar{Z}_1 = Z_3$ and thus the renormalized three-gluon coupling $G_R = g_0 Z_3{}^{3/2}/\bar{Z}_1 = g_R$.

Finally there is a Ward identity coming from Eq. (5) relating the divergence of the four-gluon vertex $\Gamma^{\mu\nu\lambda\sigma}_{ijkl}$ to the three-gluon vertex which shows that at low energies $\Gamma^{\mu\nu\lambda\sigma}_{ijkl}$ has the same behavior as in Yang-Mills theory:

$$\Gamma^{\mu\nu\lambda\sigma}_{ijkl} \to Z_4{}^{-1}\delta_{il}\delta_{jk}(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\nu}g_{\lambda\sigma} - 2g_{\nu\lambda}g_{\mu\sigma}) + 2\text{ permutations}, \tag{11}$$

and that the renormalized four-gluon vertex at zero four-momentum has strength $g_R{}^2$. Thus the low-energy structure of the renormalized Fermi theory is identical to that of Yang-Mills theory. To prove the existence of massless gluons one needs to study the Ward identity with "broken" isospin symmetry for the gluon inverse propagator:

$$\partial_\mu \left[ g_{\mu\nu}\delta(1-2)\delta_{ij} + \frac{1}{g_0}\,\frac{\delta\Gamma}{\delta A_\mu^i(1)\delta A_\nu^j(2)} \right] = \epsilon_{ilm}\left[\delta_{lj} g_{\mu\nu}\delta(1-2)\,\frac{\delta\Gamma}{\delta A_\nu^m(1)} + A_\mu^l(1)\,\frac{\delta\Gamma}{\delta A_\mu^m(1)\,\delta A_\nu^j(2)}\right]. \tag{12}$$

If, after turning off the sources, $A_\mu = \eta_\mu = \langle \overline{\psi}\gamma_\mu \tfrac{1}{2}\tau\psi\rangle \neq 0$, one has, after integrating over $\int d^4x$, that

$$\vec{\eta}_\mu \times \vec{\mathrm{D}}^{-1}{}_{\mu\nu}(k=0) = 0, \tag{13}$$

showing the existence of zero-mass bound states via a Goldstone-type theorem.

To verify these formal results one needs a calculational scheme consistent with the above Ward identities. We have explicitly verified these results to two orders in a loop expansion in the composite gluon field $A_\mu$. The loop expansion is generated by performing the Gaussian integral over the fermions in Eq. (3) and using Laplace's method to evaluate the remaining integral over the composite field $A_\mu$.[4]

We now display the renormalized Schwinger-Dyson (SD) equations for the naively divergent quantities in this theory, the boson and fermion inverse propagators and the vertex function $\Gamma_\mu$. Although the boson three- and four-point functions also naively divergent, these latter divergences are related to those of the boson inverse propagator by the Ward identities. The results presented here are a slight modification of those found for the vector-vector theory without isospin,[4,8] and the analysis of divergences in the loop expansion is similar.

The functional-differential equations for the expectation values of the field, $A_\mu{}^{\mathrm{cl}} \equiv \delta W/\delta j_\mu$, is

$$A_{\mu\,\mathrm{cl}}(x) = \overline{\psi}_{\mathrm{cl}}\gamma_\mu \tfrac{1}{2}\tau\psi_{\mathrm{cl}} + i\,\mathrm{Tr}\,\gamma_\mu\tfrac{1}{2}\tau S(xx) + j_\mu/g_0 \tag{14}$$

and for the fermion field we have

$$i\gamma\cdot\partial - m_0 + g_0\gamma_\mu\tfrac{1}{2}\tau\left(A_{\mu\,\mathrm{cl}} + \frac{1}{i}\,\frac{\delta}{\delta j_\mu}\right)\psi_{\mathrm{cl}} = -\eta. \tag{15}$$

The loop expansion follows by replacing $\delta/\delta j_\mu$ by $\epsilon\delta/\delta j_\mu$ and counting powers of $\epsilon$, and also follows from integrating over the fermions in the path integral and multiplying the effective action by $\epsilon^{-1}$.[4] The unrenormalized SD equations for the inverse propagators arise from functional differentiation of the above equations:

$$\begin{aligned} S^{-1}(12) &= (i\gamma\cdot\partial - m_0)\delta(1-2) + ig_0{}^2\gamma_\mu\tfrac{1}{2}\tau^i\int S(1Z_1)D_{ij}^{\mu\nu}(1Z_3)\Gamma_j^\nu(Z_1 2 Z_3)\,dZ_1\,dZ_3,\\ [D^{-1}]_{ij}^{\mu\nu}(12) &= \delta_{ij}g_0\delta(1-2)g_{\mu\nu} + ig_0{}^2\int \mathrm{Tr}\,\gamma_\mu\tfrac{1}{2}\tau^i S(1Z_1)\Gamma_\nu^j(Z_1Z_2 2)S(Z_2 1)dZ_1 dZ_2. \end{aligned} \tag{16}$$

$S^{-1}$ and $D^{-1}$ need two subtractions. There are two useful integral equations for $\Gamma_\lambda^\mu$ which are necessary to renormalize $S^{-1}$ and $D^{-1}$.

We have an "$s$"-channel integral equation:

$$\Gamma_i^\mu(123) = \gamma_\mu\tfrac{1}{2}\tau^i\delta(1-2)\delta(1-3) + i\int \Gamma_\nu^k(1Z_1Z_2)D_{\nu\lambda}^{kj}(Z_2Z_3)S(Z_1Z_4)[M_{2s}]_{ji}^{\lambda\mu}(Z_3Z_4, 32)\,dZ_1\,dZ_2\,dZ_3\,dZ_4, \tag{17}$$

where $M_{2s}$ is the two-particle irreducible kernel in the $s$ channel $(2+3)$ for $A_\mu$-$\psi$ scattering. The matrix $DSM$ is self-renormalizing under the multiplicative renormalization (the bar denotes renormalized quantities) $S = Z_2\overline{S}$, $D = Z_3\overline{D}$, $\Gamma_\mu = Z_1{}^{-1}\overline{\Gamma}_\mu$. Recognizing that $\Gamma^\mu$ needs one subtraction and $\Gamma^\mu(00) = \gamma_\mu\tau Z_1{}^{-1}$, one obtains the renormalized SD equation in matrix form in momentum space ($q$ is the gluon momentum):

$$\overline{\Gamma}^{\mu i}(p, p+q) = \gamma_\mu\tfrac{1}{2}\tau^i + i\{\overline{\Gamma}_\nu^k\overline{D}_{\nu\lambda}^{kj}S[M_{2s}]_{ji}^{\lambda\mu}\}_{\mathrm{sub}(1)} \tag{18}$$

where sub(1) means being subtracted once at $p = p+q = 0$. In a similar fashion, there is a "$t$"-channel integral equation

$$\Gamma_i^\mu(p, p+q) = \gamma\tfrac{1}{2}\tau^i + g_0{}^{-1}(\Lambda_{ijk}^{\mu\nu\lambda}D_{jl}^{\nu\sigma_1}D_{km}^{\lambda\sigma_2}[M_{2k}]_{lm}^{\sigma_1\sigma_2})(p, p+q) - (\Gamma^{\mu i}SSK_{2t})(p, p+q), \tag{19}$$

where $\Lambda_3$ is the three-gluon vertex, $M_{2t}$ is the 2PI kernel in the "$t$" channel for $A_\mu + A_\mu \to \psi + \overline{\psi}$, and $K_{2t}$ is the two-particle irreducible amplitude for $\psi + \overline{\psi} \to \psi + \overline{\psi}$. In its renormalized form, one has schematically

$$\overline{\Gamma}_i^\mu = \gamma_\mu\tfrac{1}{2}\tau^i + \{\overline{\Lambda}_3\overline{D}\overline{D}\overline{M}_{2t}g_R{}^{-1} - \overline{\Gamma}\overline{S}\overline{S}\overline{K}_{2t}\}_{\mathrm{sub}(1)}^{\mu i}. \tag{20}$$

These expressions for $\Gamma^\mu$ allow one to replace the *bare* vertex $\gamma_\mu \frac{1}{2}\tau$ in the integral equations for $S^{-1}$ and $[D^{-1}]^{\mu\nu}$ by $\Gamma^\mu$ minus corrections, which removes all the $Z_1$ dependence in the renormalized SD equations. Specifically $S^{-1}$ requires two subtractions. Defining $Z_2^{-1} = [\partial S^{-1}/\partial \not{p}]_{\not{p}=m} m_f = m_0 + g_0^2 \Sigma_f(\not{p} = m_f)$ where $\Sigma_f$ is the fermion self-energy, one obtains for the renormalized propagator

$$\bar{S}^{-1} = \not{p} - m_f + i g_R^2 \{\bar{\Gamma}^\mu \bar{D}^{\mu\nu} \bar{S} \bar{\Gamma}^\nu - \bar{\Gamma}^\mu \bar{D} \bar{S} \bar{M}_{2s} \bar{D} \bar{S} \Gamma^\nu\}_{\mathrm{sub}(2)}, \tag{21}$$

where we have subtracted twice at $\not{p} = m_f$. The gluon propagator, $D^{-1}{}_{\mu\nu} = g_0 \delta_{ij} g_{\mu\nu} + i g_0^2 \Sigma_{\mu\nu}(q^2)$ requires two subtractions. The Goldstone theorem [Eq. (13)] guarantees $g_0 \delta_{ij} g_{\mu\nu} = -i g_0^2 \Sigma_{\mu\nu}^{ij}(0)$ and the Ward identity ensures transversality of $\Sigma_{\mu\nu}$. Defining the boson self-energy $\Sigma$ via $\Sigma_{\mu\nu}(q^2) - \Gamma_{\mu\nu}(0) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Sigma_{\mathrm{sub}(1)}(q^2)$ and $Z_3^{-1} = i g_0^2 \Sigma_{\mathrm{sub}(1)}(0)$, one finds

$$\bar{D}^{-1}{}_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu} q^2) + i g_R^2 \{\bar{\Gamma} \bar{S} \bar{S}\, \Gamma - \bar{\Gamma} \bar{S} \bar{S} \bar{K}_{2t} \bar{S} \bar{S} \bar{\Gamma} - \frac{1}{g_R} \bar{\Lambda}_3 \bar{D} \bar{D} \bar{M}_{2t} \bar{S} \bar{S} \Gamma\}^{\mu\nu}_{\mathrm{sub}(2)}, \tag{22}$$

where

$$[g_R^2]^{-1} = \frac{1}{q^4} q_\mu q_\nu q_\lambda q_\sigma \frac{\partial}{\partial q_\lambda} \frac{\partial}{\partial q_\sigma} \left\{ \bar{\Gamma} \bar{S} \bar{S} \bar{\Gamma} - \bar{\Gamma} \bar{S} \bar{S} \bar{K}_{2t} \bar{S} \bar{S} \Gamma - \frac{1}{g_R} \bar{\Lambda}_3 \bar{D} \bar{D} \bar{M}_{2t} \bar{S} \bar{S} \bar{\Gamma} \right\}^{\mu\nu}_{\mathrm{sub}(1)} (q^2 = 0). \tag{23}$$

If we expand the integral equation for $[g_R^2]^{-1}$ in a power series in $g_R^2$ we find that each term has a logarithmic divergence. If we require that $g_R^2$ exist and be different from zero, then the coefficient of the logarithmic divergence vanishes, which is just the Johnson-Baker-Willey eigenvalue condition for $g_R^2$.[9]

We conclude that the four-fermion theory of self-coupled isospin currents is ultraviolet renormalizable in a mean-field expansion. It is characterized by one coupling-constant parameter and the fermion mass in the case of spontaneous symmetry breaking. Clearly a realistic theory of weak interactions involves the introduction of axial currents, scalar interactions among the fermions, and more complicated group structure. Even so, the basic features discussed in this note will survive.

The authors would like to thank R. Pearson and R. Dashen for valuable discussions. This work was supported in part by the U. S. Department of Energy.

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 545 (1961).
[2] N. J. Snyderman and G. S. Guralnik, in *Quark Confinement and Field Theory* (Wiley, New York, 1977), p. 13.
[3] T. Eguchi, Phys. Rev. D 14, 2755 (1976).
[4] C. Bender, F. Cooper, and G. Guralnik, Ann. Phys. (N.Y.) 109, 165 (1977).
[5] K. Tamvakis, Ph.D. thesis, Brown University, 1978 (unpublished).
[6] K. Kikkawa, Prog. Theor. Phys. 56, 947 (1976).
[7] H. Terazawa, Y. Chikashige, and K. Akama, Phys. Rev. D 15, 480 (1977).
[8] The vector-vector model without isospin has been studied in detail by J. D. Bjorken, Ann. Phys. (N.Y.) 24, 174 (1963); I. Bialynicki-Birula, Phys. Rev. 130, 465 (1963); G. S. Guralnik, Phys. Rev. 136, 1404 (1964).
[9] K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967).