where

$$Q_0 = -y(5x^2+1), \quad Q_1 = 2x(3x^2+6y^2+1), \quad Q_2 = -4y(11x^2+2y^2+1), \quad Q_3 = 8x(5x^2+12y^2+1). \tag{12}$$

Higher k's repeat in this case also. All $\beta^{(k)}$'s, $k \geq 4$, can be reduced to multiples of $\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}$.

The TS solution previously known corresponds to the restrictions $\beta^{(0)} = -12\beta^{(2)}$, $\beta^{(1)} = \beta^{(3)} = 0$. The complete solution we will now be able to generate from $\delta = 2$ will contain five arbitrary parameters including the mass. When the NUT parameter is excluded, we will have a four-parameter asymptotically flat metric. As an example, we give the solution corresponding to $\beta^{(0)} = -4\beta^{(2)}$, $\beta^{(1)} = \beta^{(3)} = 0$:

$$P_{01} = \frac{4ix(x^2-1) - 4\beta y(x^2-y^2)}{(x+1)^2(x^2-1) - 2i\beta y(x+1)(x^2-2x+y^2) - \beta^2(x^2-y^2)^2}. \tag{13}$$

The complete $\delta = 2$ solution plus further details will be published elsewhere.[5] This work was supported by the National Science Foundation under Grant No. PHY76-12246.

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Yang-Lee Edge Singularity and φ^3 Field Theory

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The edge of the gap in the distribution of Yang-Lee zeros at $H = iH_0(T)$ on the imaginary magnetic field axis in ferromagnets above T_c is essentially a critical point. In terms of the edge exponents δ and η, the density of zeros obeys $\mathcal{G}(H'') \sim [H'' - H_0(T)]^{\sigma}$, with $\sigma = 1/\delta = (d-2+\eta)/(d_2 - \eta)$. Classical behavior ($\sigma = \frac{1}{2}$) occurs for $d > d^{\times} = 6$. The appropriate field-theoretic renormalization group entails a $w\varphi^3$ coupling and, with $\epsilon = 6 - d \geq 0$, yields $\eta \approx -\epsilon/9$ for all $n < \infty$. This correlates well with refined series estimates for $d = 2$ and $d = 3$ and with exact results for $d = 1$ ($\eta = -1$).

Consider the magnetization, $M(H,T)$, of a ferromagnet at fixed temperature T. According to Yang and Lee,[1] the analytic behavior of $M(H,T)$ as a function of the magnetic field H can be understood by studying the asymptotic distribution of the zeros of the partition function in the complex magnetic-field plane $(H', H'') = (\mathrm{Re}[H], \mathrm{Im}[H])$. Very generally, the distribution of zeros in the thermodynamic limit is expected to concentrate on curvilinear loci in the complex field plane; indeed, for a variety of models of a ferromagnet,[2-4] including Ising models[2] and spherical models,[4] it is known rigorously that the zeros concentrate only on the imaginary axis, $H = iH''$. In that case the magnetization for all real and complex H can be expressed as an integral over $\mathcal{G}(H''; T)$, the asymptotic density of zeros. Below the critical temperature T_c, one has $\mathcal{G}(0;T) > 0$ and the magnetization as a function of real field exhibits a first-order transition with a jump $2M_0(T) \propto \mathcal{G}(0;T)$.

On the other hand, for $T > T_c$, there is a *gap* of width $2H_0(T)$ in the distribution of zeros, and $M(H,T)$ is analytic for $|\mathrm{Im}(H)| < H_0(T)$. The edges of this gap, at $H = \pm iH_0(T)$, must be branch points of the function $M(H,T)$; Kortman and Griffiths[5] have pointed out the interest in determining the nature of these branch points, which we term the Yang-Lee edge singularities. Since these are the signularities closest to the real axis, they play a dominant role in determining the observable behavior of M for real H and T. Indeed they should enter into the asymptotic equation of state near the critical point, although it transpires that none of the equations proposed for $d < 4$ in the current literature contain the correct singularities!

More concretely, suppose the density of zeros varies as

$$\mathcal{G}(H'') \sim |H'' - H_0(T)|^{\sigma}, \tag{1}$$

where $H'' \to H_0+$; then the magnetization exhibits a branch point of the form $m \sim h^{\sigma}$ with

$$m = M - M(iH_0, T) \text{ and } h = H - iH_0(T). \tag{2}$$

The value of the exponent σ, its universality, its dependence on the dimensionality, d, and the symmetry number, n, of the ferromagnet, and its relationship, if any, to critical exponents describing the real, directly observable singularities, are the topics of this Letter.

It will be argued that the edge singularities are closely analogous to ordinary critical points and that corresponding scaling laws and exponent relations, for example,

$$\sigma = \frac{1}{\delta} = \frac{d-2+\eta}{d+2-\eta}, \tag{3}$$

are applicable. This can be checked explicitly for the $d=1$ Ising model where $\sigma = -\frac{1}{2}$, for all $T > T_c = 0$; however, a consideration of mean-field or Landau theory indicates that the appropriate critical point is that associated with a φ^3, rather than with the usual φ^4 theory. This fact leads to a crossover dimensionality $d^{\times} = 6$, above which the classical mean-field value $\sigma = +\frac{1}{2}$ applies. A field-theoretic renormalization-group treatment is then possible and yields

$$\eta = -\tfrac{1}{9}\epsilon \text{ and } \sigma = \tfrac{1}{2} - \tfrac{1}{12}\epsilon, \tag{4}$$

to first order in $\epsilon = 6 - d$ (≥ 0); in second order the correction factor for η is $1 + a\epsilon$, with $a = \frac{43}{81}$.

More generally the values of the exponents at the edge singularity are seen to be independent of the symmetry number, or number of components, n, of the original order parameter, $\vec{s}$, provided that $n < \infty$. The limit $n \to \infty$ corresponds to the spherical model where, in fact, $\sigma = +\frac{1}{2}$ applies for all d.[4] (The nonuniformity of the $n \to \infty$ limit for $d < 6$ is puzzling and not yet understood.) The variation of σ and η with d according to this analysis is shown in Fig. 1. The solid bars for $d=2$ and 3 correspond to new high-temperature–series–expansion estimates for Ising models which extend and refine previous results of Kortman and Griffiths[5] for the square ($d=2$) and tetrahedral ($d=3$) lattices (shown by [ ]). The numerical estimates evidently accord well with the analytical expectations.

To develop the arguments, consider first the universality of σ for $T > T_c$: This is certainly to

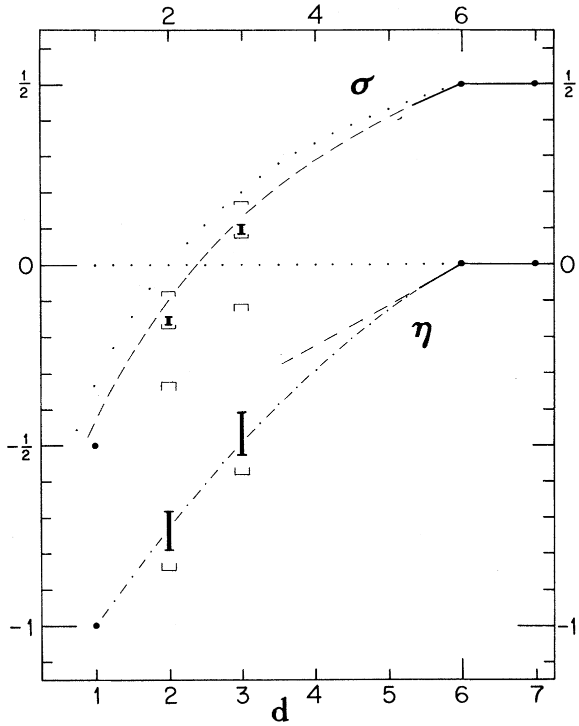

FIG. 1. Variation of the Yang-Lee edge exponents σ and η with dimensionality d. (i) The heavy black dots and solid lines correspond to exact and renormalization-group results [see Eq. (4)]; (ii) the light dotted and broken (dashed) curves represent the approximations $\eta = 0$ and $\eta = -\frac{1}{9}\epsilon$, with $\epsilon = 6-d$, inserted in Eq. (3); (iii) the dashed-dotted curve results from the two-point Padé approximant $\eta \simeq -5\epsilon/(45-4\epsilon)$, based on Eq. (4) and $\eta(1) = -1$; (iv) the solid bars for $d=2$ and $d=3$ represent new numerical estimates derived from high-temperature–series analysis for a range of lattices; (v) the more widely spaced error limits indicate the estimates of Kortman and Griffiths for the square and tetrahedral lattices (Ref. 5).

be expected on heuristic grounds. In addition, it may be checked explicitly for (i) the nearest-neighbor Ising chain,[2] and, using matrix or integral kernel methods,[6] for (ii) one-dimensional models of general n ($<\infty$); and, for all d, in (iii) spherical models ($n = \infty$),[4] and (iv) mean-field models.[5] Kortman and Griffiths[5] have also checked the numerical constancy of σ to within ± 0.05 for T not too close to T_c, for the square and tetrahedral Ising lattices. Finally, for all $n < \infty$, universality follows within the renormalization group, at least for small $\epsilon = 6-d$, from the relative stability of the fixed point describing the edge singularity.

At the critical point in zero field, however, the gap in the distribution of zeros vanished and one must expect σ to take on a different value, say σ^0. Indeed (2) then yields $M \sim H^{\sigma^0}$ so that $\sigma^0 = 1/\delta^0$, where δ^0 denotes the standard critical exponent for the critical isotherm (with $\delta^0 = 15$ for $d=2, n=1$, and $\delta^0 = 3$ for $d>4$, all n). That $\sigma^0 \neq \sigma$ may likewise be checked in all the cases (i) to (iv) above.

The idea that the edge singularity should be regarded merely as a critical point occurring at a complex or imaginary magnetic field is natural if, following Kortman and Griffiths,[5] one notes that (2) implies that the susceptibility $\chi = \partial M/\partial H$ diverges as $1/h^{1-\sigma}$ when $h = H - iH_0 \to 0$, provided that $\sigma < 1$ (which seems generally true). By analytic continuation of the pair correlation function, $G(\mathbf{R}; H, T)$, it follows that the correlation length also diverges, say as $\xi \sim 1/h^{\nu_c}$, and that $G(\mathbf{R})$ decays slowly *at* the edge singularity, say with exponent η. Standard heuristic scaling arguments[7] then suggest

$$G(\mathbf{R}; H, T) = \langle \vec{s}_0 \cdot \vec{s}_{\mathbf{R}} \rangle - \langle \vec{s}_0 \rangle^2 \approx D(Rh^{\nu_c})/R^{d-2+\eta}, \tag{5}$$

as h, $1/R \to 0$. Indeed for (i) the linear-chain Ising model, this is precisely confirmed with exponents $\eta = -1, \nu_c = \frac{1}{2}$, and scaling function $D(w) = Ae^{-Bw}/w^2$. Likewise d-dependent, hyperscaling arguments[7] lead to the exponent relation (3) and to $\nu_c = 2/(d+2-\eta)$, both of which are verified by the values quoted.

Now it is known that hyperscaling relations like (3) generally fail at borderline or crossover dimensionality, $d^\times$, and become inequalities [with the second "=" in (3) replaced by "<"] for $d > d^\times$.[8] Furthermore, one may determine $d^\times$ heuristically[9] by inserting the classical or mean-field values and solving for d. Putting $\eta = 0$, on general grounds, and[5] $\sigma = \frac{1}{2}$ (see also below) in (3) yields $d^\times = 6$. This is confirmed by renormalization-group analysis.

To set up a field-theoretic renormalization group it is essential to understand the classical or Landau phenomenological theory and to recognize that the desired critical behavior, namely a divergent susceptibility, must be sought at a complex value of the magnetic field. However, introduction of any magnetic field, real or complex, breaks the original O(n) symmetry, which means that the singular behavior (at least for $n < \infty$) should be the same as for $n = 1$ or Ising-like systems. Integral kernel methods enable one to check this directly[6] when $d = 1$. Accordingly, we may take scalar spins, $s_{\mathbf{R}}$, and consider the standard reduced Hamiltonian density[10]

$$\overline{\mathcal{H}}_{\mathbf{R}} = Hs_{\mathbf{R}} - \tfrac{1}{2}rs_{\mathbf{R}}^2 - \tfrac{1}{2}e(\nabla s_{\mathbf{R}})^2 - us_{\mathbf{R}}^4, \tag{6}$$

in which, in contrast to the usual analysis,[10] the real parameter $r \simeq C(T - T_c)$ is to be held *fixed* and *positive.*

For the classical, fluctuationless theory, set $e = 0$ and, to find a divergent susceptibility, put $s_{\mathbf{R}} \Rightarrow \bar{s}_{\mathbf{R}} + m_0$. A pure imaginary shift, $m_0 = \frac{1}{2}i(r/3u)^{1/2}$, eliminates the quadratic term in (6) and generates a cubic term $-w\bar{s}_{\mathbf{R}}^3$ with $w = 2iu(r/3u)^{1/2}$. With $m = \langle s_0 \rangle - m_0$ and $H_0(T) = u(r/3u)^{3/2}$, the equation of state is then seen to be $h \equiv (H - iH_0) \approx 3wm^2$. The residual fourth-order term, $u\bar{s}_{\mathbf{R}}^4$, plays no role asymptotically. Thus χ diverges like $h^{-1/2}$ as $h \to 0$ and so $\sigma = \frac{1}{2}$.

These considerations demonstrate that, for a field-theoretic calculation,[10] in Fourier space, it suffices to consider

$$\overline{\mathcal{H}} = h\hat{s}_0 - \tfrac{1}{2}\int_{\mathbf{q}} (r + eq^2)\hat{s}_{-\mathbf{q}}\hat{s}_{\mathbf{q}} - w\int_{\mathbf{q}}\int_{\mathbf{q}'} \hat{s}_{\mathbf{q}}\hat{s}_{\mathbf{q}'}\hat{s}_{-\mathbf{q}-\mathbf{q}'}, \tag{7}$$

with $r \equiv 0$ and $e \equiv 1$. However, one must expect w to be purely imaginary at the appropriate fixed point.[11] Of course, additional fourth-order terms are required for stabilization, but these turn out to be irrelevant for $d \geq 6$ and small ϵ. It is now straightforward to use perturbation theory in w to construct a momentum-shell integration renormalization group[10] with a spatial rescaling $\mathbf{q} \Rightarrow \mathbf{q}'/b$. A shift and spin rescaling, $\hat{s}_{\mathbf{q}} \Rightarrow \hat{m}\delta_{\mathbf{q}} + \hat{c}\hat{s}'_{\mathbf{q}'}$, are required to maintain $r' = r \equiv 0$ and $e' = e \equiv 1$. (Note that one more condition is imposed than at a normal critical point.) In zeroth order one finds $\hat{c}^2 \approx b^{d+2+O(w^2)}$ so that $\eta = O(w^2)$ which leads to the recursion relations $h' \approx b^{(d/2)+1}h$ and $w' \approx b^{\epsilon/2}w$ with $\epsilon = 6 - d$. This confirms $d^\times = 6$ as the borderline dimensionality.

A first-order calculation, involving graphs with one, two, and three third-order vertices, yields $\eta \approx 6K_6 w^2$ and a fixed point $h^* = \frac{3}{4}K_6 q_\Lambda^4 w^*$, and $w^* = i\epsilon^{1/2}/(54K_6)^{1/2}$ (where q_Λ is the cutoff momentum and K_6 is the area of a unit sphere at $d = 6$). These results imply $\eta \approx -\epsilon/9$ [see (4)]; the hyperscaling relation (3) is checked directly by the eigenvalue $\lambda_1 = 1/\nu_c \approx 4 - \frac{4}{9}\epsilon \equiv \frac{1}{2}(d + 2 - \eta)$.

An alternative renormalization-group procedure imposes $h' = h \equiv 0$ and allows r to vary in (7). This leads to completely equivalent physical results although some surprisingly singular renormalization-group flows are encountered. It is also possible[12] to adapt Amit's dimensional regularization calculation[11] by relaxing his Potts-model constraint [and putting $\alpha_1=\beta_1=\beta_4=Q_{111}\neq 0$ in his Eqs. (6.1)–(6.3) *et seq.*]. This confirms (4) and yields the second-order term quoted.

Finally, as demonstrated by Kortman and Griffiths,[5] the exponent σ may be estimated numerically for Ising models by studying the high-temperature series for χ with purely imaginary field and varying temperature. However, if the temperature independence of σ is accepted, more accurate results can be found by studying the high-temperature limit of the series. If one sets $z=\tanh(J/k_BT)[\tanh(H/k_BT)]^2$, where J is the nearest-neighbor coupling, one finds[13] that the expansions then reduce to those for the monomer-dimer problem[14] on the same lattice at dimer activity z. It follows that the Yang-Lee zeros for the dimer problem must lie on the negative z axis[15] and the dimer density, $\rho(z)$, must exhibit a branch point of the form $\rho_0+A(z-z_0)^\sigma$. Long series are available for the monomer-dimer problem[14,16] on various lattices. From ratio analysis[7] of expansion coefficients for the square (17 terms), triangular (14 terms), tetrahedral (16 terms), sc (15 terms), bcc (13 terms), and fcc (10 terms) lattices the following estimates have been obtained:

$$\sigma=-0.155\pm 0.010 \quad \text{for } d=2,$$
$$\sigma=0.098\pm 0.012 \quad \text{for } d=3. \tag{8}$$

Within their quoted uncertainties of ± 0.050, these compare well with the Kortman-Griffiths estimates[5] of $\sigma\simeq-\frac{1}{8}$ $(d=2)$, $+\frac{1}{8}$ $(d=3)$; see Fig. 1. Independent estimates for the square and triangular lattices differ by about 0.006; likewise, in three dimensions lattices of larger coordination number suggest values 0.006–0.012 higher. These differences, however, are within the extrapolation uncertainties; so there are no serious grounds for doubting that σ is truly lattice independent. Padé-approximant analyses of various logarithmic derivative series[6] yield consistent results.

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