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Group Transformation That Generates the Kerr and Tomimatsu-Sato Metrics

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For stationary axially symmetric vacuum metrics, we give a series of transformations $\beta^{(k)}$ which automatically preserve asymptotic flatness. We show how to generate the Kerr metric from the Schwarzschild, using $\beta^{(0)}$. We also show, using $\beta^{(k)}$, that the Tomimatsu-Sata (TS) class of metrices must be larger than previously realized, and for $\delta=2$ there is a five-parameter TS metric. As an example, we present a two-parameter metric from this family, which we claim to be a new, physically realistic, asymptotically flat, rotating vacuum solution.

In previous papers[1-3] we have studied the Einstein vacuum equations for stationary axially symmetric gravitational fields. We have found that equations are presented by an infinite-dimensional symmetry group K. The transformations of K are labeled $\gamma_{AB}{}^{(k)}$, real and symmetric, $A, B=1, 2$, $k=0, \pm1, \pm2, \ldots$. The $\gamma_{AB}{}^{(k)}$ act upon a sequence of complex potentials $N_{AB}{}^{(m,n)}$, $A, B=1, 2$, $m=0, 1, \ldots,$ $n=1, 2, \ldots$ which characterize a given space-time. The transformations may be used to generate new solutions from old ones. However, since each of the $\gamma_{AB}{}^{(k)}$ violates asymptotic flatness, the new solutions thus produced are not physically interesting.

We have now found that the commuting subgroup of transformations

$$\beta^{(k)}=\gamma_{22}{}^{(k+2)}+\gamma_{11}{}^{(k)} \tag{1}$$

leaves Minkowski space invariant. Hence a space-time which is asymptotically Minkowskian is guaranteed to remain so under these $\beta^{(k)}$ transformations. A great wealth of new and interesting vacuum solutions can thus be generated, and we have just begun to explore the possibilities. For example we have found that when $\beta^{(0)}$ is applied to the Schwarzschild metric, the metric generated is Kerr.

The $\gamma_{AB}{}^{(k)}$ transformations are given for infinitesimal values of the parameter by

$$\gamma_{AB}{}^{(k)}: N_{AB}{}^{(m,n)} \to N_{AB}{}^{(m,n)} + \gamma_{AX}{}^{(k)} N^{X}{}_{B}{}^{(m+k,n)} + \gamma_{XB}{}^{(k)} N_{A}{}^{X(m,n+k)} + \gamma^{XY(k)} \sum_{s=1}^{k} N_{AX}{}^{(m,s)} N_{YB}{}^{(k-s,n)} \tag{2}$$

(where indices are raised using ϵ_{AB}). The $\beta^{(k)}$ transformations are more conveniently expressed in terms of P_{mn}, which are certain linear combinations of $N_{AB}{}^{(m,n)}$:

$$P_{0n}=N_{11}{}^{(0n)}+iN_{12}{}^{(0,n-1)},$$
$$P_{mn}=N_{11}{}^{(m,n)}-iN_{21}{}^{(m-1,n)}+iN_{12}{}^{(m,n-1)}+N_{22}{}^{(m-1,n-1)}, \quad m>0. \tag{3}$$

[Note that

$$P_{01}=-i(\mathcal{E}-1)$$

where $\mathcal{E}$ is the usual Ernst potential.] The infinitesimal version of $\beta^{(k)}$ is

$$\beta^{(k)}: P_{0n} \to P_{0n} + \beta^{(k)}\Big(-2iP_{0,n+k+1} + \sum_{s=1}^{k+2} P_{0s} P_{k+2-s,n}\Big),$$
$$P_{mn} \to P_{mn} + \beta^{(k)}\Big(2iP_{m+k+1,n} - 2iP_{m,n+k+1} + \sum_{s=1}^{k+2} P_{ms}P_{k+2-s,n}\Big). \tag{4}$$

The transformations may be applied to any stationary axially symmetric space-time. First we must calculate the potentials P_{mn} for this space-time (a laborious but straightforward process). Equation (4) may then be iterated to give a power series in $\beta^{(k)}$. To the best of our knowledge, the series cannot always be summed. However, in many cases of interest, the simplicity of the P_{mn} will permit us to obtain the result in closed form.

Members of the Kerr-Tomimatsu-Sato class of solutions are particularly good candidates for the initial metric. If the distortion parameter δ is an integer, we have shown[4] that there exists a gauge in which $P_{mn} = 0$ when $m > \delta$ or $n > \delta$. It would be difficult to work in this gauge, however, since it does not obey Eqs. (2.18) and (2.22) of Ref. 2. (Those conditions were used in deriving the group transformations, and they could not be removed now without making extensive changes.) Fortunately, a simple modification of this gauge leads to one which does satisfy our conditions.

For the Schwarzschild metric, we use the potentials

$$P_{01} = 2i/(x+1), \quad P_{11} = 4iy/(x+1),$$
$$P_{2r,2s} = 0, \quad P_{2r+1,2s+1} = \binom{2r}{r}\binom{2s}{s}P_{11}, \tag{5}$$
$$P_{2r+1,2s} = A_{rs}, \quad P_{2r,2s+1} = \binom{2r}{r}\binom{2s}{s}P_{01} + A_{rs},$$

where A_{rs} are certain numerical constants, e.g., $A_{10} = 0$, $A_{01} = 4i$. The array P_{mn} effectively repeats itself, and the only independent functions it contains are P_{01}, P_{11}.

When $\beta^{(0)}$ is applied to this array, the infinite set of transformation equations all turn out to be repetitions of just two:

$$P_{01} \to P_{01} + \beta P_{01}P_{11}, \quad P_{11} \to P_{11} + \beta(P_{11}P_{11} + 8iP_{01} + 16). \tag{6}$$

They may be regarded as a pair of differential equations,

$$\partial P_{01}/\partial\beta = P_{01}P_{11}, \quad \partial P_{11}/\partial\beta = P_{11}P_{11} + 8iP_{01} + 16, \tag{7}$$

whose solutions are

$$P_{01} = \frac{2i}{a\cos 4\beta - ib\sin 4\beta + 1}, \quad P_{11} = 4i\left(\frac{b\cos 4\beta - ia\sin 4\beta}{a\cos 4\beta - ib\sin 4\beta + 1}\right), \tag{8}$$

where a, b do not depend on β. From the initial conditions at $\beta = 0$, we find $a = x$, $b = y$. The reparametrization

$$p = \cos 4\beta, \quad q = -\sin 4\beta \tag{9}$$

puts P_{01} in agreement with the usual Ernst potential for the Kerr metric.

When $\beta^{(1)}$ is applied to the Schwarzschild potentials, the transformation equations reduce to

$$P_{01} \to P_{01} + 4\beta(P_{01}P_{01} + 4\beta(P_{01}P_{01} - 2iP_{01}). \tag{10}$$

This can be shown to generate NUT (Newman-Unti-Tamburino) space. Higher k's repeat, with even k's producing Kerr and odd k's producing NUT.

Similar results can be obtained for $\delta = 2$. For infinitesimal β we find

$$P_{01} \to \frac{4ix}{(x+1)^2} + \frac{16}{(x+1)^4}\sum_k \beta^{(k)}Q_k, \tag{11}$$

where

$$Q_0 = -y(5x^2+1), \quad Q_1 = 2x(3x^2+6y^2+1), \quad Q_2 = -4y(11x^2+2y^2+1), \quad Q_3 = 8x(5x^2+12y^2+1). \tag{12}$$

Higher k's repeat in this case also. All $\beta^{(k)}$'s, $k \geq 4$, can be reduced to multiples of $\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}$.

The TS solution previously known corresponds to the restrictions $\beta^{(0)} = -12\beta^{(2)}$, $\beta^{(1)} = \beta^{(3)} = 0$. The complete solution we will now be able to generate from $\delta = 2$ will contain five arbitrary parameters including the mass. When the NUT parameter is excluded, we will have a four-parameter asymptotically flat metric. As an example, we give the solution corresponding to $\beta^{(0)} = -4\beta^{(2)}$, $\beta^{(1)} = \beta^{(3)} = 0$:

$$P_{01} = \frac{4ix(x^2-1) - 4\beta y(x^2-y^2)}{(x+1)^2(x^2-1) - 2i\beta y(x+1)(x^2-2x+y^2) - \beta^2(x^2-y^2)^2}. \tag{13}$$

The complete $\delta = 2$ solution plus further details will be published elsewhere.[5] This work was supported by the National Science Foundation under Grant No. PHY76-12246.

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Yang-Lee Edge Singularity and φ^3 Field Theory

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The edge of the gap in the distribution of Yang-Lee zeros at $H = iH_0(T)$ on the imaginary magnetic field axis in ferromagnets above T_c is essentially a critical point. In terms of the edge exponents δ and η, the density of zeros obeys $\mathcal{G}(H'') \sim [H'' - H_0(T)]^\sigma$, with $\sigma = 1/\delta = (d-2+\eta)/(d_2 - \eta)$. Classical behavior ($\sigma = \frac{1}{2}$) occurs for $d > d^\times = 6$. The appropriate field-theoretic renormalization group entails a $w\varphi^3$ coupling and, with $\epsilon = 6 - d \geq 0$, yields $\eta \approx -\epsilon/9$ for all $n < \infty$. This correlates well with refined series estimates for $d = 2$ and $d = 3$ and with exact results for $d = 1$ ($\eta = -1$).

Consider the magnetization, $M(H,T)$, of a ferromagnet at fixed temperature T. According to Yang and Lee,[1] the analytic behavior of $M(H,T)$ as a function of the magnetic field H can be understood by studying the asymptotic distribution of the zeros of the partition function in the complex magnetic-field plane $(H', H'') = (\mathrm{Re}[H], \mathrm{Im}[H])$. Very generally, the distribution of zeros in the thermodynamic limit is expected to concentrate on curvilinear loci in the complex field plane; indeed, for a variety of models of a ferromagnet,[2-4] including Ising models[2] and spherical models,[4] it is known rigorously that the zeros concentrate only on the imaginary axis, $H = iH''$. In that case the magnetization for all real and complex H can be expressed as an integral over $\mathcal{G}(H''; T)$, the asymptotic density of zeros. Below the critical temperature T_c, one has $\mathcal{G}(0;T) > 0$ and the magnetization as a function of real field exhibits a first-order transition with a jump $2M_0(T) \propto \mathcal{G}(0; T)$.

On the other hand, for $T > T_c$, there is a *gap* of width $2H_0(T)$ in the distribution of zeros, and $M(H,T)$ is analytic for $|\mathrm{Im}(H)| < H_0(T)$. The edges of this gap, at $H = \pm iH_0(T)$, must be branch points of the function $M(H,T)$; Kortman and Griffiths[5] have pointed out the interest in determining the nature of these branch points, which we term the Yang-Lee edge singularities. Since these are the signularities closest to the real axis, they play a dominant role in determining the observable behavior of M for real H and T. Indeed they should enter into the asymptotic equation of state near the critical point, although it transpires that none of the equations proposed for $d < 4$ in the current literature contain the correct singularities!

