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Differential Form of Real-Space Renormalization: Exact Results for Two-Dimensional Ising Models

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A new real-space renormalization method is developed which leads to differential renormalization-group equations. For a two-dimensional triangular Ising lattice these equations can be solved exactly and yield and exact critical properties.

The distinguishing feature of the renormalization-group (RG) approach[1] to problems in statistical mechanics is that it immediately focuses upon the critical properties of a system. These properties appear as the solution of the RG recursion relations, which in general may take either a differential form or the form of a discrete transformation. The RG approach contrasts sharply with traditional methods in which critical properties play no special role whatever. Unfortunately, RG recursion relations obtained by the real-space method[2] can be solved only in certain special cases. Examples are some one-dimensional models, or models constructed especially to be amenable to RG treatment.[3]

The real-space renormalization method has, until now, always led to discrete renormalization transformations. We present here a novel real-space renormalization procedure which leads to a differential formulation of the RG. We show how exact differential RG equations can be obtained for the two-dimensional triangular Ising model. We solve these equations exactly for the critical properties, and find agreement with the results obtained by Onsager[4] and Houtappel.[5]

To initiate an infinitesimal renormalization procedure we consider the RG equation relating the original Hamiltonian $\mathcal{H}(s)$ and the renormalized one $\mathcal{H}'(s')$,

$$\exp[\mathcal{H}'(s')] = \frac{\mathrm{Tr}_s \exp[\mathcal{H}_I(s',s) + \mathcal{H}(s)]}{\mathrm{Tr}_{s'} \exp[\mathcal{H}_I(s',s)]}, \tag{1}$$

where $\mathcal{H}_I(s',s)$ couples the two systems. If we impose upon $\mathcal{H}_I$ the special requirement that

$$\mathrm{Tr}_{s'} \exp[\mathcal{H}_I(s',s)] = \exp[\mathcal{H}(s)], \tag{2}$$

then it follows from Eq. (1) that

$$\mathrm{Tr}_s \exp[\mathcal{H}_I(s',s)] = \exp[\mathcal{H}'(s')]. \tag{3}$$

Clearly if $\mathcal{H}_I$ were symmetric under the interchange of s and s', Eq. (1) would simply be an identity transformation and there would be no flow in Hamiltonian space, nor any information to be obtained from the RG. If, however, $\mathcal{H}_I$ fails infinitesimally from being symmetric in s and s', then $\mathcal{H}'(s')$ will differ infinitesimally from $\mathcal{H}(s)$ and a flow in Hamiltonian space is generated.

We illustrate this idea for a system of Ising spins. To construct $\mathcal{H}_I(s',s)$ we couple two triangular-lattice systems that differ infinitesimally in number of spins. The unprimed lattice has a lattice constant a and has the shape of an equilateral triangle with side of length L. Hence there are $L/a+1$ lattice sites along each side. The primed lattice is a similar array but with L/a sites along each side (see Fig. 1). Each spin of one lattice is coupled to its nearest-neighbor spins on the other lattice by couplings $p_i(\vec{R})$. Both for these couplings and for the interactions $K_i(\vec{R})$ to be introduced shortly we shall let the index i correspond to an orientation in space, as shown in Fig. 1. We shall label each p_i and K_i by the coordinate $\vec{R}$ corresponding to the center of the upward-pointing triangle of the unprimed lattice to which the bond belongs. The coupling Hamiltonian $\mathcal{H}_I(s', s)$ is taken as a sum of terms of the form $(p_1s_1 + p_2s_2 + p_3s_3)s_1'$, where we omit the coordinate arguments. Equation (2) relates the coupling constants $p_i(\vec{R})$ in a triangle of the unprimed lattice to the interaction constants $K_j(\vec{R})$ $(i,j = 1, 2, 3)$ in that triangle by a star-triangle transformation.[6] Its explicit form is

$$K_i(\vec{R}) = F(p_i(\vec{R}), p_j(\vec{R}), p_k(\vec{R})), \quad i,j,k \text{ cyclic}, \tag{4a}$$

$$F(p_i,p_j,p_k) = \tfrac{1}{4}\ln[\cosh(p_i+p_j+p_k)\cosh(-p_i+p_j+p_k)/\cosh(p_i-p_j+p_k)\cosh(p_i+p_j-p_k)]. \tag{4b}$$

Equation (3) relates the couplings p_i to interactions $\tilde{K}_j$ between the primed spins by a similar star-triangle transformation, differing from Eq. (4) only in that p_1, p_2, and p_3 enter with unequal coordinate arguments. We shall make this explicit. If $\vec{R}$ is the center of the unprimed up triangle in Fig. 2, then the interaction $\tilde{K}_i$ of the primed up triangle shown is constructed from three couplings p_j $(j = 1, 2, 3)$ with coordinates differing from $\vec{R}$ by small displacement vectors $a\vec{\rho}_{ij}$. In order to express the $\vec{\rho}_{ij}$ in explicit form we introduce unit vectors $\hat{x}$ and $\hat{y}$ and define (see Fig. 1)

$$\vec{e}_1 = -\hat{y}/\sqrt{3}, \quad \vec{e}_2 = \hat{x}/2 + \hat{y}/2\sqrt{3}, \quad \vec{e}_3 = -\hat{x}/2 + \hat{y}/2\sqrt{3}. \tag{5}$$

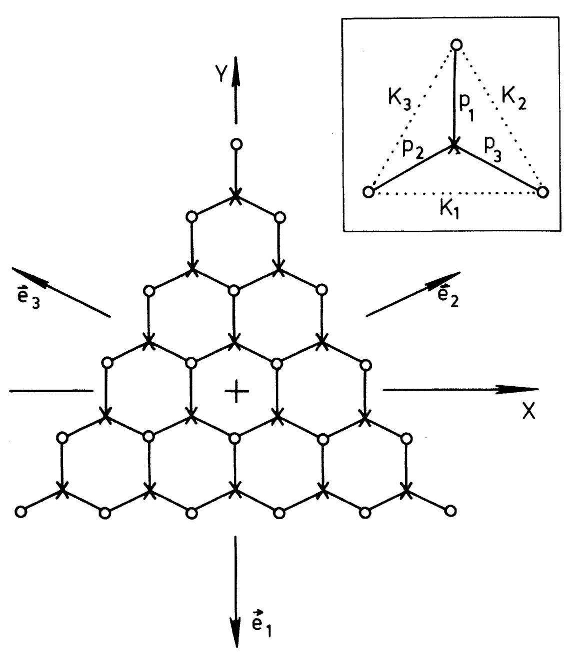

FIG. 1. The couplings p_i connect the sites of the unprimed triangular lattice (circles) to those of the primed lattice (crosses). Inset: The correspondence between the index i and the orientations of the couplings p_i and interactions K_i.

We then have $\vec{\rho}_{ij} = \vec{e}_i + \vec{e}_j - \hat{y}/\sqrt{3}$. Hence,

$$\tilde{K}_i(\vec{R} - \tfrac{1}{3}\sqrt{3}a\hat{y}) = F(p_i(\vec{R}+a\vec{\rho}_{ii}), p_j(\vec{R}+a\vec{\rho}_{ij}), p_k(\vec{R}+a\vec{\rho}_{ik})), \tag{6}$$

where i,j,k are cyclic. The coordinate $\vec{R} - \frac{1}{3}\sqrt{3}a\hat{y}$ is the center of the upward triangle in the primed lattice to which the $\tilde{K}_i$ refer.

We shall now consider the spatial coordinates as continuous variables. If $p_i(\vec{R})$ varies only over distances of order L, then we can Taylor expand Eq. (6) to obtain

$$\tilde{K}_i(\vec{R} - \tfrac{1}{3}\sqrt{3}a\hat{y}) = K_i(\vec{R}) + a\textstyle\sum_l Q_{il}\vec{\rho}_{li}\cdot\nabla p_l + O(a^2/L^2), \tag{7}$$

where $Q_{il} = \partial F(p_i,p_j,p_k)/\partial p_l$ with i, j, k cyclic. The gradient of p_l in Eq. (7) is easily related to the gradient of K_m by $\nabla p_l = \sum_m Q^{-1}{}_{lm}\nabla K_m$, where $Q^{-1}{}_{lm}$ is an element of the inverse matrix. In order to derive from Eq. (7) a renormalization transformation we rescale the coordinates of the $\tilde{K}_i$ in such a way that their range becomes the

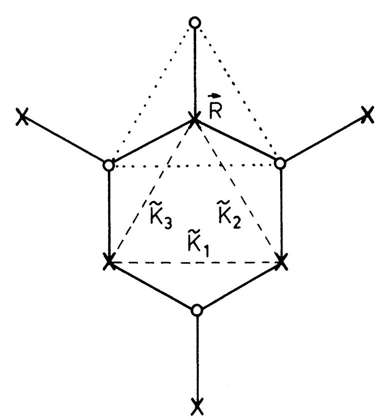

FIG. 2. The bonds p_i (solid lines) involved in the calculations of the interactions $\tilde{K}_i$ (dashed lines). Also shown is a reference triangle of the unprimed lattice (dotted lines), centered at $\vec{R}$.

same as that of the coordinates of the K_j. That is, we put $\vec{R}' \equiv [(L+a)/L](\vec{R} - \frac{1}{3}\sqrt{3}a\hat{y})$ and define renormalized couplings K_i' by $K_i'(\vec{R}') = \tilde{K}_i(\vec{R} - \frac{1}{3}\sqrt{3}a\hat{y})$. We use these definitions in Eq. (7) and expand $K_i'(\vec{R}')$ about $\vec{R}$. Putting $\vec{r} = \vec{R}/L$, $\delta K_i = K_i'(\vec{R}) - K_i(\vec{R})$, and $\delta t = a/L$, we obtain for $a/L \to 0$ the renormalization equations

$$\partial K_i/\partial t = \sum_j \vec{D}_{ij} \cdot \nabla K_j - \vec{r} \cdot \nabla K_i, \quad i = 1, 2, 3, \qquad (8)$$

where ∇ now differentiates with respect to $\vec{r}$, and $\vec{D}_{ij} \equiv \sum_k (\vec{e}_i + \vec{e}_k) Q_{ik} Q^{-1}{}_{kj}$. The equations have to be solved in an equilateral triangle with side of length one and center in the origin, for given initial condition at $t = 0$, and subject to the appropriate boundary conditions. To see what the boundary conditions are, we note that the renormalized interaction between two spins on the border of the primed lattice arises from a star-triangle transformation in which one of the couplings p_i has vanished. Hence in order that the renormalized couplings along the sides of the lattice be given by the same expression (8), we have to impose that $p_1 = 0$, $p_2 = 0$, and $p_3 = 0$, along the lower, the right, and the left borders of the triangular domain, respectively.

In this Letter we limit ourselves to the study of the critical properties of Eq. (8). The flow generated by Eq. (8) has an important invariance property, which is most easily described with the aid of the functions $u_i = \sinh 2K_j \sinh 2K_k$ (i, j, k cyclic). One can verify that the *subspace of functions $K_i(\vec{r})$ that satisfy locally (i.e. for every $\vec{r}$) the "criticality condition" $\sum_i u_i(\vec{r}) = 1$ is left invariant by the flow.* We note that the relation $\sum_i u_i = 1$ is precisely Houtappel's condition[5] for the critical surface in $K_1K_2K_3$ space of a homogeneous triangular lattice. If we introduce the normal vector $\xi_j = \partial \sum_i u_i / \partial K_j$, then the invariance property can be stated as $\sum_j \xi_j(\vec{r}) \partial K_j(\vec{r})/\partial t = 0$ whenever $K_j(\vec{r})$ lies in the critical subspace. This result is most easily verified by using Eq. (8) and expressing functions of the $K_i(\vec{r})$ in terms of the $p_i(\vec{r})$.

Within the invariant subspace one can locate a critical fixed-point solution $K_i^*(\vec{r})$, defined by $\partial K_i^*/\partial t = 0$. In terms of the u_i we find

$$u_i^*(\vec{r}) = \tfrac{1}{3} - 2\vec{r} \cdot \vec{e}_i, \quad i = 1, 2, 3, \qquad (9)$$

which satisfies the boundary conditions and has triangular symmetry. While the lattice described by K_i^* is isotropic triangular in the center, it deforms continuously and is square near the vertices, and one-dimensional with infinitely strong couplings along the sides.

Finally by putting $K_i(\vec{r}) = K_i^*(\vec{r}) + \psi_i(\vec{r})$ and linearizing Eq. (8) about $K_i^*(\vec{r})$, we obtain a linear flow problem of the form $\partial \psi_i(\vec{r})/\partial t = \sum_j T_{ij}^*(\vec{r}, \nabla)\psi_j(\vec{r})$. In this approach the critical exponents are identical to the eigenvalues of the operator T^*. We expect the fixed-point solution to be stable within the critical subspace, but unstable in directions away from it. The analysis of the eigenvalues in the unstable (temperaturelike) directions is facilitated by our knowledge of the invariant subspace. It follows that the adjoint operator $\tilde{T}^*$ has temperaturelike eigenfunctions of the form $\varphi_i(\vec{r}) = f(\vec{r})\xi_i^*(\vec{r})$, where ξ_i^* is the normal vector evaluated at K_j^*. Upon substitution, a scalar eigenfunction equation for f results. After considerable algebra, again facilitated by working with the functions $p_i(\vec{r})$, this equation simplifies to

$$f(\vec{r}) = y f(\vec{r}). \qquad (10)$$

Hence there is an eigenvalue $y = 1$, in accordance with the exact result.[4,5] It is infinitely degenerate and any $f(\vec{r})$ is an eigenfunction. In fact Eq. (10) is a demonstration of universality, since the fixed-point solution (9) contains locally different critical systems which by (10) all have the exponent $y = 1$.

In conclusion we have derived an exact real-space RG recursion in differential form for the two-dimensional triangular Ising model with three different spatially dependent nearest-neighbor interactions $K_i(\vec{r})$. We have solved these equations without approximation for the critical properties, and found exact results for both the critical temperatures and the temperaturelike critical singularities. Further details will be published elsewhere.

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[1] K. G. Wilson, Phys. Rev. B 4, 3174 (1971); S.-k. Ma, Rev. Mod. Phys. 45, 589 (1973); K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974); M. E. Fish-

er, Rev. Mod. Phys. 46, 597 (1974); see also *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.

[2]See the review by Th. Niemeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.

[3]P. M. Bleher and Ja. G. Sinai, Commun. Math. Phys. 33, 23 (1973); S. Krinsky and D. Furman, Phys. Rev. Lett. 32, 731 (1974); T. L. Bell and K. G. Wilson, Phys. Rev. B 11, 3431 (1975); D. R. Nelson and M. E. Fisher, Ann. Phys. (N.Y.) 91, 226 (1975); Th. Niemeyer and Th. W. Ruijgrok, Physica (Utrecht) 81A, 427 (1975); R. G. Priest, Phys. Rev. B 11, 3461 (1975); M. Nauenberg, J. Math. Phys. 16, 703 (1975); H. J. Hilhorst, J. Stat. Phys. 17, 413 (1977).

[4]L. Onsager, Phys. Rev. 65, 117 (1944).

[5]R. M. F. Houtappel, Physica (Utrecht) 16, 425 (1950). See also G. F. Newell, Phys. Rev. 79, 876 (1950).

[6]See Ref. 5 and also G. F. Newell and Elliott W. Montroll, Rev. Mod. Phys. 25, 353 (1953); I. Syozi, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 1.

Group Transformation That Generates the Kerr and Tomimatsu-Sato Metrics

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For stationary axially symmetric vacuum metrics, we give a series of transformations $\beta^{(k)}$ which automatically preserve asymptotic flatness. We show how to generate the Kerr metric from the Schwarzschild, using $\beta^{(0)}$. We also show, using $\beta^{(k)}$, that the Tomimatsu-Sata (TS) class of metrices must be larger than previously realized, and for $\delta=2$ there is a five-parameter TS metric. As an example, we present a two-parameter metric from this family, which we claim to be a new, physically realistic, asymptotically flat, rotating vacuum solution.

In previous papers[1-3] we have studied the Einstein vacuum equations for stationary axially symmetric gravitational fields. We have found that equations are presented by an infinite-dimensional symmetry group K. The transformations of K are labeled $\gamma_{AB}{}^{(k)}$, real and symmetric, $A, B=1, 2$, $k=0, \pm1, \pm2, \ldots$. The $\gamma_{AB}{}^{(k)}$ act upon a sequence of complex potentials $N_{AB}{}^{(m,n)}$, $A, B=1, 2$, $m=0, 1, \ldots,$ $n=1, 2, \ldots$ which characterize a given space-time. The transformations may be used to generate new solutions from old ones. However, since each of the $\gamma_{AB}{}^{(k)}$ violates asymptotic flatness, the new solutions thus produced are not physically interesting.

We have now found that the commuting subgroup of transformations

$$\beta^{(k)}=\gamma_{22}{}^{(k+2)}+\gamma_{11}{}^{(k)} \tag{1}$$

leaves Minkowski space invariant. Hence a space-time which is asymptotically Minkowskian is guaranteed to remain so under these $\beta^{(k)}$ transformations. A great wealth of new and interesting vacuum solutions can thus be generated, and we have just begun to explore the possibilities. For example we have found that when $\beta^{(0)}$ is applied to the Schwarzschild metric, the metric generated is Kerr.

The $\gamma_{AB}{}^{(k)}$ transformations are given for infinitesimal values of the parameter by

$$\gamma_{AB}{}^{(k)}\!:\!N_{AB}{}^{(m,n)}\rightarrow N_{AB}{}^{(m,n)}+\gamma_{AX}{}^{(k)}N^{X}{}_{B}{}^{(m+k,n)}+\gamma_{XB}{}^{(k)}N_{A}{}^{X(m,n+k)}+\gamma^{XY(k)}\sum_{s=1}^{k}N_{AX}{}^{(m,s)}N_{YB}{}^{(k-s,n)} \tag{2}$$

(where indices are raised using ϵ_{AB}). The $\beta^{(k)}$ transformations are more conveniently expressed in terms of P_{mn}, which are certain linear combinations of $N_{AB}{}^{(m,n)}$:

$$P_{0n}=N_{11}{}^{(0n)}+iN_{12}{}^{(0,n-1)},$$
$$P_{mn}=N_{11}{}^{(m,n)}-iN_{21}{}^{(m-1,n)}+iN_{12}{}^{(m,n-1)}+N_{22}{}^{(m-1,n-1)},\quad m>0. \tag{3}$$

[Note that

$$P_{01}=-i(\mathcal{E}-1)$$

