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[5]J. Lebowitz, J. Percus, and J. Sykes. Phys. Rev. 188, 487 (1967); J. Sykes: J. Stat. Phys. 8, 279 (1973).

[6]H. van Beijeren and M. Ernst, Physica (Utrecht) 68, 437 (1973), and 70, 225 (1973).

[7]As shown in Refs. 5 and 6, this modification is of second order in the gradients $[\propto(\partial/\partial\vec{r}_1)^2]$ for a simple gas; therefore, it does not affect the transport coefficients. Yet, for *arbitrary* f_1, the following is not applicable to the original Enskog equation.

[8]For an interesting approach at the level of N-body master equations, see I. Prigogine, C. George, F. Henin, and L. Rosenfeld, Chem. Scr. 4, 5 (1973).

[9]Of course, exactly as in the dilute-gas limit, if ρ_N was the exact d.f., $S(t)$ would remain constant [see, for example, S. Rice and P. Gray, *The Statistical Mechanics of Simple Liquids* (Interscience, New York, 1965)]; the use of the approximate ρ_N, Eq. (3), is just a trick to *define* $S(t)$ in terms of f_1 (or W_1); since this latter function (and not for the exact ρ_N) is an irreversible behavior *a priori* expected.

Screening Solutions to Classical Yang-Mills Theory

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We present two new solutions to the classical Yang-Mills field equations in the presence of a localized external source. These solutions totally screen the charge of the source. They have lower energy than the corresponding Coulomb solution.

Non-Abelian gauge theories offer the greatest promise to describe the elementary forces in nature. We here investigate the solutions to the classical Yang-Mills equations in the presence of a static external source in Minkowski space:

$$(D_\mu F^{\mu\nu})^a = j^{a\nu}(x) = \delta^{\nu 0} q^a(x), \tag{1a}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g c^{abc} A_\mu{}^b A_\nu{}^c, \tag{1b}$$

where $q^a(x)q^a(x)$ is time independent. By a local gauge transformation, one can always line up the source into commuting directions of color space, e.g., $q^a(x) \to \delta^{a3}[q^b(x)q^b(x)]^{1/2} = \delta^{a3}q(\vec{x})$ for SU(2) which for simplicity we will study first. The *Ansatz*[1] $A_\mu{}^a = \delta^{a3}A_\mu$ then reduces Eqs. (1) to the Maxwell equations of electrodynamics. We call the corresponding solution the Coulomb solution for the source $q^a(x)$.

However, various results in the literature have already shown that classical unbroken Yang-Mills theories in Minkowski space are qualitatively different from electrodynamics, e.g., the Wu-Yang monopole[2] and Coleman's non-Abelian plane wave[3] which are both nontrivial solutions to Eqs. (1) with $q^a(x) = 0$. Moreover, Mandula[4] has shown that the Coulomb solution corresponding to a static source distributed over a thin spherical shell is unstable if $gQ > \frac{3}{2}$, where $Q = \int d^3x [q^a(x)q^a(x)]^{1/2}$. Mandula also showed that the instability modes produce an inward flow of charge that tends to screen the external source. Since the energy is positive definite, Eqs. (1) must admit static solutions of lower energy than the Coulomb one. Below we exhibit two new types of solutions to Eqs. (1) with localized and integrable static sources. The first type has the long-range behavior of a magnetic dipole field, and has lower energy than the Coulomb solution once gQ is large enough. The second type has no long-range field strengths at all, and its energy can be made arbitrarily small.

The magnetic dipole solution.—The *Ansatz*

$$A_0{}^1 = A_0{}^2 = A_i{}^2 = A_i{}^3 = 0,$$
$$A_0{}^3 = \varphi(\rho, x_3), \quad A_i{}^1 = \epsilon_{i3j}(x_j/\rho)A(\rho, x_3), \tag{2}$$

where $\rho = [x_1{}^2 + x_2{}^2]^{1/2}$, assures that all the Eqs. (1) are automatically satisfied provided

$$-\nabla^2\varphi + g^2A^2\varphi = q, \tag{3a}$$

$$+\nabla^2 A - \rho^{-2}A + g^2\varphi^2 A = 0. \tag{3b}$$

The Coulomb solution corresponds to setting $A = 0$. Outside of this *Ansatz*, the full nonlinearity of the equations comes into play and there are no analytical methods available. It is nevertheless possible to show that there exists a whole class (a continuous infinity) of charge distributions $q(\vec{x})$ which are localized and integrable (i.e., $Q<\infty$) and which admit besides the Coulomb potential a new type of solution with $A\neq 0$ and $\varphi\neq 0$ and finite total energy. To this end, let us consider *any* field $A(\rho,x_3)$ which satisfies the following two conditions: $A(\rho,x_3)$ goes to zero as $r=[x_1^2+x_2^2++x_3^2]^{1/2}\to 0$. Away from the origin, $A(\rho,x_3)$ approaches exponentially fast the solution $\mathfrak{A}=\rho/r^3$ of $\nabla^2\mathfrak{A}-\rho^{-2}\mathfrak{A}=0$. For that given $A(\rho,x_3)$, let us successively solve Eq. (3b) for $\varphi(\vec{x})$ and calculate $q(\vec{x})$ from $\varphi(\vec{x})$, $A(\vec{x})$, and Eq. (3a). For the charge distribution $q(\vec{x})$ thus found, $\varphi(\vec{x})$ and $A(\vec{x})$ will be an exact solution of the field equations. The second condition on $A(\vec{x})$ assures that both $\varphi(\vec{x})$ and $q(\vec{x})$ vanish exponentially fast away from the origin. The first and second conditions together assure finiteness of the energy. Let us give a particular example:

$$A(\rho,x_3)=ca\,\frac{\rho}{r^3}\tanh\left(\frac{r}{a}\right)^3,$$
$$\varphi(\rho,x_3)=\frac{\sqrt{18}}{a^3 g}\,\frac{r^2}{\cosh^2[(r/a)^3]} \qquad (4)$$

are solutions of Eqs. (3) for a rather complicated but nonsingular charge distribution $q(\vec{x})$ spread over a region of width a, and of total charge

$$Q=\int d^3x\, g^2A^2\varphi=c^2gI_1, \qquad (5)$$

where

$$I_1=\frac{8\pi\sqrt{18}}{3}\int_0^\infty dx\,\frac{\tanh^2 x^3}{\cosh x^3},$$

for our particular example. The particular charge distribution we obtain depends, of course, on the particular choice we made for the way $A(\rho,x_3)$ approaches ρ/r^3 in the transition region between $r\ll a$ and $r\gg a$. The point is that to the continuous infinity of ways in which $A(\rho,x_3)$ can approach ρ/r^3 corresponds a continuous infinity of localized charge distributions which admit solutions of the new type. Presumably the thin spherical shell studied by Mandula is among these charge distributions.

The new solution has the long-range behavior of a magnetic dipole field. Indeed, using a vector notation for the spatial components, we have for $r\gg a$

$$\vec{A}^1\cong ca(\hat{3}\times\vec{x})r^{-3}=-\vec{m}\times\nabla(r^{-1}),$$
$$\vec{B}^1=\nabla\times\vec{A}^1\cong\frac{3(\vec{m}\times\vec{x})-\vec{m}r^2}{r^5}, \qquad (6)$$

where $\vec{m}=ca\hat{3}$. In Eqs. (2) and (6) the orientation of the magnetic dipole has been arbitrarily chosen to be along the 3 direction of space and the 1 direction of isospin space (it could have been any linear combination of the 1 and 2 isospin directions). The other field strengths are either zero or short range. The physical situation is as follows. The Yang-Mills fields $\vec{A}^1$ and φ create a charge distribution $-g^2(\vec{A}^1)^2\varphi$ whose total charge exactly cancels Q. The electric field strengths thus become short range. On the other hand, the Yang-Mills fields create a current loop distribution

$$\vec{j}^1=g^2\varphi^2\vec{A}^1=g^2\varphi^2A\rho^{-1}(\hat{3}\times\vec{x}), \qquad (7)$$

whose total magnetic moment is precisely $\vec{m}=ca\hat{3}$.

The energy of the magnetic dipole solution has the following form:

$$H^{\text{md}}=\int d^3x\,\tfrac{1}{2}[|\nabla\varphi|^2+g^2\varphi^2(\vec{A}^1)^2+(\nabla\times\vec{A}^1)^2]$$
$$=\int d^3x[\tfrac{1}{2}|\nabla\varphi|^2+g^2\varphi^2(\vec{A}^1)^2]$$
$$=\frac{1}{a}\left(\frac{1}{g^2}I_2+c^2I_3\right)=\frac{1}{a}\left(\frac{1}{g^2}I_2+\frac{Q}{g}\frac{I_3}{I_1}\right), \qquad (8)$$

where I_2 and I_3 (like I_1) are calculable numbers which depend on the shape of the charge distribution but not on its norm (Q) nor its spatial extension (a). Since the energy of the Coulomb solution has the general form

$$H^c=a^{-1}Q^2I_4, \qquad (9)$$

we find that the magnetic dipole solution has lower energy than the Coulomb solution when

$$Qg>\frac{1}{2I_4}\left\{\frac{I_3}{I_1}+\left[\left(\frac{I_3}{I_1}\right)^2+4I_2I_4\right]^{1/2}\right\}. \qquad (10)$$

The total screening solution.—The magnetic dipole solution thus appears to be precisely the new type of solution whose existence had been implied by Mandula's work. However, we will now show, by merely exploiting our knowledge of the Coulomb solution and gauge invariance, that any extended charge distribution admits solutions of energy as low as one wishes. Indeed, while in Abelian gauge theories the sign of a charge is unambiguously defined, this is not so in non-Abelian gauge theories where the direction in isospin space of a charge distribution can be locally reversed by a gauge transformation. The only gauge-invariant quantity that characterizes a source is $q^2(x)=q^a(x)q^a(x)$ for SU(2).[5] Thus for any given extended source $q^2(x)$, we can choose a gauge where half of the source is lined up in the positive 3 direction of isospin space and the other half in the negative 3 direction. We can then make the *Ansatz* $A_\mu^a = \delta^{a3}A_\mu$ which will yield a Coulomb solution corresponding to an electric dipole. By rotating back into the gauge where q^a is completely lined up in the positive 3 direction, we find a solution whose energy is that of a dipole field although q^a is in the monopole configuration. It is clear that from dipole we can go to quadrupole and so on, lowering the energy indefinitely in the process. Let us illustrate this by giving a particular example in which all fields will be free of discontinuities. Equations (1) are solved by

$$A_0{}^a = 0, \quad A_i{}^a = E_i{}^a t, \quad E_i{}^a = \frac{Q}{4\pi}\frac{x_i}{r^3}\frac{1}{2\pi n}\{\delta^{a2}[\cos(2\pi n h(r)) - 1] - \delta^{a3}\sin(2\pi n h(r))\},$$
$$q^a = \frac{Q}{4\pi}\left(\frac{-1}{r^2}\frac{dh}{dr}\right)\{\delta^{a2}\sin[2\pi n h(r)] + \delta^{a3}\cos[2\pi n h(r)]\}, \tag{11}$$

where t is time, n is an integer, and $h(r)$ is an arbitrary function that goes to 1 as $r \to 0$ and goes to 0 as $r \to \infty$, say $h(r) = \exp[-\frac{1}{2}(r/a)^2]$. Rotated back into the gauge where q^a is completely lined up in the positive 3 direction of isospin space, the solution has the form

$$A'_0{}^a = 0, \quad A'_i{}^a = E'_i{}^a t - \delta^{a1} g^{-1}\partial_i[2\pi n h(r)],$$
$$E'_i{}^a = \frac{Q}{4\pi}\frac{x_i}{r^3}\frac{1}{2\pi n}\{\delta^{a2}[1 - \cos(2\pi n h(r))] - \delta^{a3}\sin(2\pi n h(r))\}, \quad q'^a = \delta^{a3}\frac{Q}{4\pi}\left(\frac{-1}{r^2}\frac{dh}{dr}\right). \tag{12}$$

The electric field is completely screened because the charge distribution $-g\,\epsilon^{abc}A_i'^bE_i'^c$ carried by the Yang-Mills fields exactly cancels the external source. There is no magnetic field. The energy of this total screening solution,

$$H^{\rm ts} = \frac{Q^2}{2\pi}\left(\frac{1}{2\pi n}\right)^2\frac{1}{a}\int_0^\infty \frac{dx}{x^2}\sin^2 \pi n h(xa), \tag{13}$$

is finite provided $1 - h(r) \sim r^{1/2+E}$, with $E > 0$, as $r \to 0$ in which case $H^{\rm ts}$ goes to zero as $n \to \infty$.[6]

In conclusion, we have shown by exploiting our knowledge of the Coulomb solution and gauge invariance that the Yang-Mills field equations in the presence of a static extended external source admit solutions which completely screen the external source and which have energy as low as one wishes. But we have also shown that there is yet more structure to the Yang-Mills equations in the presence of external sources: They also admit solutions of the magentic dipole type whose energy becomes lower than that of the Coulomb solution when gQ is larger than some critical value. These solutions cannot be transformed to a Coulomb solution by any gauge transformation since $B_i{}^a \neq 0$ and $\epsilon^{abc}E_i{}^bE_j{}^c \neq 0$. The generalization of the above results to larger gauge groups is trivial only if the source lies completely within an SU(2) subgroup. This and other questions related to this work will be expanded upon in a later publication.

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[1]See, for example, H. G. Loos, Nucl. Phys. 72, 677 (1965), and references therein.

[2]T. T. Wu and C. N. Yang, in *Properties of Matter Under Unusual Conditions*, edited by K. Mack and S. Fernbach (Interscience, New York, 1969).

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[5]In general, the number of invariants that characterize a source equals the rank of the group; e.g., for SU(3), the invariants are $q^a(x)q^a(x)$ and $d_{abc}q^a(x)q^b(x)q^c(x)$.

[6]To see this, dominate $\sin^2[\pi n h(r)]$ by $\pi^2 n^2[1 - h(r)]^2$ for $0 \leq r \leq a/n$, and by 1 for $r > a/n$.